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Financial bubbles and capital accumulation in altruistic economies

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ABSTRACT

We consider an overlapping generations model à la Diamond (1965) with two additional ingredients: altruism and an asset (or land) bringing non-stationary positive dividends (or fruits). We study the global dynamics of capital stocks and asset values as well as the interplay between them. Asset price bubbles are also investigated.

1. Introduction

According to the literature on pure rational bubbles (asset without dividend) à la Tirole (1985), a bubble may coexist with physical capital because (1) agents want to buy the asset at any date (the young buys the bubble from the old) and (2) the real interest rate of the economy without bubble asset is lower than the population growth rate (*the economy experiences capital overaccumulation or low interest rate*).[1] Although this literature is huge, very few papers have tackled the issue of bubble when dividends are positive. Many unaddressed questions on bubbles with positive dividend remain. Why do these bubbles arise? What are their dynamic properties? How do the capital and financial asset values interfere over time? What is the difference between bubbles of assets with and without dividends?

Our goal is to address these open issues. In addition, we generalize Tirole (1985) with a kind of altruism. Altruism matters affecting the offspring's saving and the portfolio composition. Therefore, the novelty of the paper is twofold and rests on the introduction of forward (or descending) altruism and a financial asset (or land) bringing non-stationary positive dividends (or fruits) in the overlapping generations (OLG) benchmark à la Diamond (1965).

First, we prove that standard Inada condition ensures the existence of an interior intertemporal equilibrium. We do so in two steps: (1) proving the existence in finite-horizon cases, and (2) passing to the limit, we get an equilibrium for the infinite-horizon case. Notice that, without Inada condition, this existence result may fail. Indeed, in a low productivity situation, households prefer to invest in financial asset instead of physical capital, which may lead to zero aggregate capital (this is possible because households can consume dividends).

Results on equilibrium existence are complemented by a global analysis of equilibrium including the case of bubbly equilibria. As in the standard literature (Tirole, 1982; Kocherlakota, 1992; Santos and Woodford, 1997; Huang and Werner, 2000), we say that a bubble exists at an equilibrium if the equilibrium price of financial asset exceeds the present discounted value of its dividends, that is its *fundamental value*. In short, we call the *bubble* the difference between the asset price and the fundamental value. This equals the value at infinity of one unit of asset. In particular, when dividend is zero at any date, the asset is called *bubble* by Tirole (1985) or *fiat money* by other authors (Bewley, 1980; Weil, 1987).

We firstly prove that, if there is no bubbly equilibrium, then the economy has a unique equilibrium. Hence, the main part of our analysis focuses on multiple equilibria where bubbles may appear.

One of our main results is that a bubble exists only if the sum over time of ratios of dividend to production is finite. By consequence, in a bounded economy,[2] a bubble exists only if the sum over time of dividends is finite. This entails a number of implications. For instance, when dividends are strictly positive, there does not exist a steady stated associated with a bubble in the

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[1] The reader is referred to Brunnermeier and Oehmke (2013) for a survey on bubbles. To connect growth and pure bubbles, see Hirano and Yanagawa (forthcoming), Bosi and Pham (2016) and references therein.

[2] That is output per capita is uniformly bounded from above.

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asset; this property holds whatever the level of interest rate. By contrast, as proved by Tirole (1985), a pure bubble may arise at the steady state: this is the very difference between bubbles in assets with and without dividend. A particular case of our setup is Weil (1990) who provided an example of bubble where dividends may be positive but become zero after a finite number of periods.

We also show that, in a bounded economy with *high interest rate* (i.e., the interest rate at the steady state of the economy without financial asset is strictly higher than the population growth rate), there does not exist asset bubble. This result is independent of the level of asset dividends and, in this respect, quintessential. Of course, it covers Tirole (1985), where dividends are zero at any date, and rests on the following intuition. As seen above, in a bounded economy, bubbles are excluded when dividends do not converge to zero. When dividends converge to zero, we can prove that in the long run (1) the capital stock is bounded from above by that at the steady state of the economy without financial asset and (2) the asset value converges to zero. Combining these properties and the high interest rate condition, the discounted value of one unit of asset converges to zero, which means that there is no bubble.

Summing up, we obtain two necessary conditions for bounded economies, under which bubble may arise: (1) a low interest rate and (2) a finite sum of dividends. Interestingly, we prove that along a bubbly equilibrium, capital stocks converge either to the steady state of the economy without financial asset or to the level at which the interest rate equals the rate of population growth. This implies in turn that asset values must converge along a bubbly equilibrium.

Our above general findings are complemented by analyses in special cases. More precisely, in the case of Cobb–Douglas and linear technologies, we obtain a continuum of bubbly equilibria. Closed forms are also computed under some specifications. We find that a higher degree of forward altruism lowers the interest rate in the economy without financial asset. In this respect, we can say that descendent altruism promotes bubbles. To the best of our knowledge, these examples are the first ones dealing with bubble of an long-lived asset having positive dividends in a production economy with concave technology.

In the last part of the paper, we revisit the connection between bubble, interest rate and asset price. The seminal article by Tirole (1985) finds out that existence of pure bubbles requires a low interest rate. Such conclusion rests on the boundedness of aggregate output, including asset dividends. Indeed, in the case of high interest rate, if a bubble exists, the asset values grow to infinity and the equilibrium feasibility is violated. However, we argue that, in the case of unbounded growth (of the capital-free side of production), incomes of households are high enough to cover the value of asset with bubble (that agents may buy) even if this asset value grows to infinity (because of high interest rate). Moreover, in such an economy, dividends are no longer required to be bounded. This is also an added value of our paper.

At a first sight, we may be convinced that asset prices increase in time along a bubbly equilibrium. However, we provide a counterexample of bubbly equilibrium along which asset prices may increase, decrease or even fluctuate in time. This means that there is no robust causal link between bubble existence and monotonicity of asset prices.

The rest of the paper is organized as follows. Section 2 introduces the economic fundamentals. Sections 3 and 4 present some equilibrium properties and a formal definition of bubble. Section 5 provides general results on equilibrium transition for bubbles and capital. Section 6 and Section 7 focus on particular cases and global dynamics. All the technical proofs are gathered in Appendices.

2. Model

We consider a two-period OLG model of rational bubbles in the spirit of Diamond (1965), Tirole (1985) and Weil (1987). Time is discrete $t = 0, 1, 2, \ldots$.

Production. At each date, there is a representative firm with the production function $F(K, L)$ where K and L are the aggregate capital and the labor forces. We require standard assumptions.

Assumption 1. F is constant returns to scale, concave, strictly increasing and in C^2.

Let R_t and w_t represent the return on capital and the wage rate. Profit maximization under complete capital depreciation implies

$$R_t = R(k_t) \equiv f'(k_t) \text{ and } w_t = w(k_t) \equiv f(k_t) - k_t f'(k_t) \tag{1}$$

where $k_t \equiv K_t/L_t$ denotes the capital intensity, $f(k_t) \equiv F(k_t, 1)$.

Generations. Assume that there are N_t new individuals entering the economy at time t. The growth factor of population is supposed to be constant: $n = N_{t+1}/N_t$.

Households. Each young agent lives for two periods and supplies one unit of labor. Assume that preferences of households are rationalized by an additively separable utility function

$$U(c_t, d_{t+1}) \equiv u(c_t) + \beta u(d_{t+1})$$

where β represents the degree of patience, while c_t and d_{t+1} denote the consumption demands at time t and $t+1$ of a household born at time t.

Assumption 2. u is in C^2, $u'(c) > 0 > u''(c)$, $u'(0) = \infty$.

Agent born at date t saves through a portfolio (a_t, s_t) of financial asset and physical capital. Consumption prices are normalized to one. q_t and $\delta_t \geq 0$ denote the asset price and the dividend in consumption units, while

$$b_t \equiv q_t a_t \text{ and } \xi_t \equiv \delta_t a_t$$

the values of asset and dividend respectively. The sequence of dividends (δ_t) is assumed to be exogenous.

Once households buy the asset a_t, they will be able to resell it tomorrow and perceive dividends (in terms of consumption good). This asset can also be interpreted as a Lucas' tree or land, or stock as in Kocherlakota (1992).

Budget constraints of household born at date t are written

$$c_t + s_t + q_t a_t \leq w_t + g_t \tag{2}$$

$$d_{t+1} + n g_{t+1} \leq R_{t+1} s_t + (q_{t+1} + \delta_{t+1}) a_t \tag{3}$$

$$x d_{t+1} \leq n g_{t+1} \tag{4}$$

where g_{t+1} represents the bequests from parents to offspring and x is the degree of forward (or descending) altruism.

There are two theoretical approaches to bequests. (1) In the case of selfish preferences, households leave only unintended bequests due to lifespan uncertainty (Davies, 1981) or leave bequests to receive care in the old age and give more to the child who provides more care. (2) In the case of altruistic preferences, households leave bequests to offspring even if children provide no care and give more to the child with greater needs (Becker, 1981).

Empirical studies show that bequests matter. Kotlikoff and Summers (1981) calculate the share of intergenerational transfers in total households' wealth in the United States and find a range between 46% and 81% according to the method used. Other studies show lower shares. About two-thirds of the studies using U.S. data support the altruism model while those using French data support the selfish exchange model (Laferrere and Wolff, 2006).

Our model is a model of altruistic preferences. Instead of considering as in Barro (1974) the utility of children in the utility of parents, we introduce a "moral" constraint (which can be interpreted either as naive behavior or the result of social pressures (either moral or religious)): parents leave a share of their wealth when old to offspring. In a two-period OLG model this wealth coincides with the second-period consumption. A commitment to leave a given fraction is more observable than a choice based on the utility of offspring in the utility of parents (Barro and Becker, 1989). Our model is justified on the empirical ground because, as seen above, bequests matter and many empirical studies support an altruistic behavior. Moreover, this kind of altruism allows us to have a tractable model.[3] [4]

The market clearing conditions sum up to $N_t s_t = K_{t+1}$ and $N_t a_t = N_{t+1} a_{t+1}$, that is, respectively to

$$s_t = nk_{t+1} \text{ (capital)} \tag{5}$$

$$a_t = na_{t+1} \text{ (financial asset)}. \tag{6}$$

Definition 1. Let $k_0 > 0, g_0 > 0$ be given. A positive list $(q_t, R_t, w_t, c_t, d_{t+1}, g_{t+1}, s_t, a_t, k_{t+1})_t$ is an intertemporal equilibrium for the economy with forward altruism if (i) given $(q_t, q_{t+1}, R_t, w_t, g_t)$, the allocation $(c_t, d_{t+1}, g_{t+1}, s_t, a_t)$ maximizes $U(c_t, d_{t+1})$ subject to constraints (2), (3), (4) and (ii) conditions (5), (6) are satisfied for any $t \geq 0$.

Under Inada condition $f'(0) = \infty$, we have $k_t > 0$ for any t.[5] So, in the rest of the paper we will focus on equilibria with $k_t > 0$ for any t. In this case, the consumer's program leads to an (equilibrium) no-arbitrage condition:

$$q_t = \frac{q_{t+1} + \delta_{t+1}}{R_{t+1}} \tag{7}$$

meaning that what we pay to buy 1 unit of asset today equals what 1 unit of asset will bring for us tomorrow.

Remark 1. At equilibrium, the budget constraints become binding. Combining them with (5), (6) and (7) we obtain a sequence $(b_t, k_{t+1})_{t\geq 0}$ which is a reduced and equivalent form of equilibrium. Thus, from now on, we will refer to this sequence as an equilibrium.

3. Equilibrium

This section provides some basic equilibrium properties and introduces the notion of bubble.

Constraints (2), (3) and (4) entail $ng_t = xd_t$. Combining this with (6) and (7), we observe that the household's total saving $s_t + b_t$ only depends on $w_t + g_t$ and R_{t+1}. Moreover, since the function u is strictly concave, the solution of household problem is unique and we can write

$$nk_{t+1} + b_t = \mathcal{S}_x(w_t + g_t, R_{t+1}) \tag{8}$$

where $\mathcal{S}_x$ is interpreted as a saving function. We require the following assumption under which the function $\mathcal{S}_x$ is increasing in R_{t+1} (see De la Croix and Michel (2002) for instance).

Assumption 3. The function $cu'(c)$ is increasing.

Since (3), (4) are binding, we obtain $g_t = xd_t/n$ and $d_t(1+x) = R_t nk_t + (q_t + \delta_t)a_{t-1}$. Combining this with (6), we get that

$$g_t = \frac{x}{1+x}\left(k_t f'(k_t) + b_t + \xi_t\right). \tag{9}$$

By consequence, Eq. (8) becomes

$$nk_{t+1} + b_t - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_t f'(k_t) + \frac{x}{1+x}(b_t + \xi_t), f'(k_{t+1})\Big) = 0.$$

Remark 2. Conditions (2), (3), (4) and (9) imply that

$$c_t + nk_{t+1} + \frac{b_t}{1+x} = f(k_t) - \frac{k_t f'(k_t)}{1+x} + \frac{x\xi_t}{1+x} \tag{10}$$

and hence $b_t \leq (1+x)f(k_t) + x\xi_t$. Therefore, if k_t and ξ_t are bounded from above, the asset value will be also bounded from above.

We can summarize as follows.

Lemma 1. *Let $k_0 > 0, g_0 > 0$ be given. Assume that Assumptions 1–3 hold. Then, the sequence $(k_{t+1}, b_t)_{t\geq 0}$ is an interior equilibrium if and only if*

$$nk_1 + b_0 - \mathcal{S}_x\Big(w_0 + g_0, f'(k_1)\Big) = 0 \tag{11}$$

$$nk_{t+1} + b_t - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_t f'(k_t) + \frac{x}{1+x}(b_t + \xi_t), f'(k_{t+1})\Big) = 0 \quad \text{for } t \geq 1 \tag{12}$$

$$b_{t+1} = b_t \frac{f'(k_{t+1})}{n} - \xi_{t+1} \quad \text{for } t \geq 0 \tag{13}$$

$$b_t > 0, \quad k_{t+1} > 0 \quad \text{for } t \geq 0. \tag{14}$$

Moreover, the system (11)–(14), in the case it has solution, is equivalent to (11), (13), (14) and

$$k_{t+1} = \mathcal{G}_x(k_t, b_t, \xi_t) \tag{15}$$

where the function $\mathcal{G} : \mathbb{R}^3_+ \to \mathbb{R}$ is defined as the solution of

$$H_{k_t,b_t,\xi_t}(k) \equiv nk + b_t - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_t f'(k_t) + \frac{x}{1+x}(b_t + \xi_t), f'(k)\Big) = 0.$$

$\mathcal{G}_x$ is continuously differentiable,

$$\frac{\partial \mathcal{G}_x}{\partial k_t} > 0, \ \frac{\partial \mathcal{G}_x}{\partial b_t} < 0, \ \frac{\partial \mathcal{G}_x}{\partial \xi_t} > 0.$$

and $\mathcal{G}_x(k, 0, \xi) > 0$ for any $k > 0$ any $\xi > 0$.

Proof. See Appendix A.1. □

All assumptions in Lemma 1 are for instance satisfied with Cobb–Douglas production function $F(K, L) = AK^{\alpha}L^{1-\alpha}$ and isoelastic preferences $U(c, d) = \ln c + \beta \ln d$ or $U(c, d) = \left(c^{1-\sigma} + \beta d^{1-\sigma}\right)/(1-\sigma)$ with $\sigma \in (0, 1)$.

It should be noticed that we need to prove the existence of solution of the system (11)–(14) before having the recursive equation (15). In the following, we will present the existence of solution of the system (11)–(14) which is essential to explore equilibrium properties. Before doing this, it is natural to impose the following assumption.

Assumption 4. $\bar{\xi} \equiv \sup_t \xi_t < \infty$ or, equivalently, $\sup_t \left(\delta_t a_0/n^t\right) < \infty$.

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[3] See Bosi et al. (2016) for bubbles in an OLG model where altruism à la Barro (1974) is introduced through a recursive utility.

[4] The reader is referred to Michel et al. (2006) for a review of altruism and Galperti and Strulovici (forthcoming) for an axiomatic theory of intergenerational altruism.

[5] Sections 6.2 and 7 provide equilibrium properties for the linear technology case.

Lemma 2 (*Existence of an Interior Equilibrium*). *Given k_0, g_0. Assume that Assumptions 1, 2 holds and the function $cu'(c)$ is increasing. If $f'(0) = \infty$, then the system (11)–(14) has a solution $(b_t, k_{t+1})_{t\geq 0}$ and such a sequence is an interior equilibrium.*

Proof. See Appendix A.2. □

Comment. This existence result is far from trivial. One may think that it can be easily proved by the following argument: given $b_0 > 0$, any $(k_t, b_t)_{t\geq 1}$ determined by (11)–(13) is an equilibrium; so, there are multiply equilibria. However, this argument is not correct because $(k_t, b_t)_{t\geq 1}$ determined only by (11)–(13) may be negative at some date. The point is to prove that there exists $b_0 > 0$ such that the sequence $(k_t, b_t)_{t\geq 1}$ determined by (11)–(13) is positive.

The existence of an interior equilibrium rests on a sufficiently high productivity of capital ($f'(0) = \infty$). This equilibrium may fail to exist in the case of low productivity. An example of failure with linear technology is provided in Section 6.2 and supplemented with economic interpretation.[6]

Lemma 2 is a generalized version of Proposition 1.2 in De la Croix and Michel (2002) where they prove the equilibrium existence in an OLG model as in our framework but without financial asset. Their proof cannot be directly applied in our model because of the presence of the long-lived asset with non-stationary dividends. Our proof consists of two steps: (1) proving the existence in finite-horizon cases, and (2) passing to the limit, we get an equilibrium for the infinite-horizon case. The reader is referred to Citanna and Siconolfi (2010, 2012) for the generic existence of a recursive equilibrium in stochastic OLG economies. It seems that their results cannot be directly applied to our framework because we consider a production economy with a long-lived asset having non-stationary dividends.

4. Definition and existence of bubbles

In this section, we present a formal definition of bubble and a characterization of bubble existence as its direct consequence.

Solving recursively (7), we obtain an asset price decomposition in two parts

$$q_t = Q_{t,t+\tau} q_{t+\tau} + \sum_{s=1}^{\tau} Q_{t,t+s}\delta_{t+s}, \text{ where}$$
$$Q_{t,t+s} \equiv \frac{1}{R_{t+1}\dots R_{t+s}}$$

is the discount factor of the economy from date t to $t + s$.

In the spirit of Tirole (1982), Tirole (1985), Kocherlakota (1992), Santos and Woodford (1997) and Huang and Werner (2000), we define the fundamental value of financial asset and the bubble.

Definition 2.

1. The Fundamental Value of a unit of asset at date t is the sum of discounted values of dividends:
$$FV_t \equiv \sum_{s=1}^{\infty} Q_{t,t+s}\delta_{t+s}.$$
2. We say that there is a bubble at date t if $q_t > FV_t$.
3. When $\delta_t = 0$ for any $t \geq 0$ (the Fundamental Value is zero), we say that there is a pure bubble if $q_t > 0$ for any t.

[6] See Le Van and Pham (2016) for equilibrium analysis in an infinite-horizon general equilibrium model where the aggregate capital stock k_t may be zero.

Clearly, we have $q_t = FV_t + \lim_{\tau\to\infty} Q_{t,t+\tau}q_{t+\tau}$. Thus, condition $q_t - FV_t > 0$ does not depend on t. Therefore, if a bubble exists at date 0, it exists forever. Moreover, we also see that $q_{t+1} - FV_{t+1} = R_{t+1}(q_t - FV_t)$.

Remark 3. Our asset is related to the asset with rent (dividend) in Tirole (1985) since both the assets bring dividends at any date. However, Tirole (1985) assumes that the rent (dividend) is stationary while dividends are non-stationary in our model. In Tirole (1985), there is no bubble with a positive rent, while, in our model, asset bubbles may arise as we will show below.

Weil (1990) considers an asset (he calls land) with positive dividends, but in a pure exchange economy, and he assumes that there exists t_0 such that $\delta_t = 0$ for any $t \geq t_0$, while our model encompasses the productive sector and δ_t may be strictly positive at any date.

It should be noticed that when $\delta_t = 0$ for any t, some others, e.g. Weil (1987) or Bewley (1980), interpret the asset as *fiat money*.

For notational simplicity, we set $Q_0 \equiv 1$ and $Q_t \equiv Q_{0,t}$ for any t. No-arbitrage condition (7) implies that

$$q_0 = \frac{1}{R_1}(1+\frac{\delta_1}{q_1})q_1 = \frac{1}{R_1R_2}(1+\frac{\delta_1}{q_1})(1+\frac{\delta_2}{q_2})q_2 = \cdots$$
$$= Q_T q_T(1+\frac{\delta_1}{q_1})\cdots(1+\frac{\delta_T}{q_T}).$$

Bubbles exist if and only if $\lim_{T\to\infty} Q_T q_T > 0$ which is equivalent to $\prod_{t=1}^{\infty}(1+\delta_t/q_t) < \infty$. It is easy to see that $\prod_{t=1}^{\infty}(1+\delta_t/q_t) < \infty$ holds if and only if $\sum_{t=0}^{\infty}\delta_t/q_t < \infty$. Therefore, we have necessary and sufficient conditions (based on endogenous variables) for the existence of bubbles of assets with positive dividends.[7]

Proposition 1. *In the case of strictly positive dividends ($\delta_t > 0$ for any t), the following statements are equivalent.*[8]

1. *A bubble exists at date t.*
2. $\lim_{T\to\infty} Q_T q_T > 0$, *i.e.* $\lim_{T\to\infty} b_T n^T / \Pi_{\tau=1}^{T} f'(k_\tau) > 0$.
3. $\sum_{t=0}^{\infty}\delta_t/q_t < \infty$, *i.e.* $\sum_{t=0}^{\infty}\xi_t/b_t < +\infty$.

Proposition 1 is very general because its proof rests only on the no-arbitrage condition (7) and Definition 2. Here, technology and preferences play no role.

Let us give another interpretation of bubble condition $\sum_{t=0}^{\infty}\delta_t/q_t < \infty$. Look at budget constraints

$$c_t + s_t + q_t a_t \leq w_t + g_t$$
$$d_{t+1} + n g_{t+1} \leq R_{t+1}s_t + (q_{t+1} + \delta_{t+1})\, a_t.$$

We may rewrite

$$(q_{t+1} + \delta_{t+1})a_t = q_{t+1}\Big(1 + \frac{\delta_{t+1}}{q_{t+1}}\Big)a_t.$$

Here, one buys a_t units of asset at date t, with price q_t. At the next date (date $t + 1$), she receives $(1 + \delta_{t+1}/q_{t+1})a_t$ units of the same asset, with price q_{t+1}. By the way, δ_{t+1}/q_{t+1} can be interpreted as the financial asset's interest rate (in terms of asset, not in terms of consumption good) between dates t and $(t + 1)$. So, bubble condition $\sum_{t=0}^{\infty}\delta_t/q_t < \infty$ may be named "*low asset interest rates condition*".

Remark 4 (*No-arbitrage Condition Revisited*). The above interpretation allows us to revisit the no-arbitrage condition (7) which can be rewritten as $R_{t+1}q_t/q_{t+1} - 1 = \delta_{t+1}/q_{t+1}$. Let $\tau_{t+1} \equiv q_{t+1}/q_t - 1$

[7] These conditions are similar to those in Montrucchio (2004) or Le Van and Pham (2014).

[8] Condition $\delta_t > 0\ \forall t$ is to ensure that $q_t > 0$ at any date, which is needed to define δ_t/q_t.

the inflation rate calculated with the asset prices. We also defined $r_{t+1} \equiv R_{t+1} - 1$. We then obtain:

$$\frac{1 + r_{t+1}}{1 + \tau_{t+1}} - 1 = \frac{\delta_{t+1}}{q_{t+1}}. \tag{16}$$

By approximating $\frac{1+r_{t+1}}{1+\tau_{t+1}} \approx (1 + r_{t+1} - \tau_{t+1})$, we obtain so-called no-arbitrage condition:

$$r_{t+1} \approx \frac{\delta_{t+1}}{q_{t+1}} + \tau_{t+1}. \tag{17}$$

This means that the real return in terms of consumption good (which is the numeraire) in the production sector equals the sum of the interest rate (in terms of asset) and the inflation rate calculated with the asset prices.

We now come back with the issue of asset bubbles. By combining Proposition 1 and Remark 2, we can prove an important consequence: bubble existence requires very low dividends with respect to output.

Corollary 1. *Consider the case of positive dividends ($\delta_t > 0$ for any t).*

Bubble existence implies $\sum_{s=1}^{\infty} [\xi_s / f(k_s)] < \infty$. Consequently, if $\xi_t = \xi$ $\forall t$, there does not exist a steady state associated with a bubble in the asset.

Comments.

1. Notice that Corollary 1 does not require any condition about the boundedness of capital stock or dividends. It also holds for non-stationary technologies.
   Corollary 1 is stronger than a well-known result of literature on rational bubbles in infinite-horizon models (Le Van and Pham, 2014, 2016): bubbles are ruled out if the sequence of ratio of dividend to aggregate output is bounded below from zero.
2. Let us interpret the asset a_t as land and ξ_t as fruits of land at period t. Thanks to Corollary 1, we realize why Weil (1990) needs to assume that trees produce fruits only for a finite number of periods[9] in order to get land bubbles.
3. *Bubbles of assets with and without dividends*. The last point of Corollary 1 means that, at the steady state, an asset yielding positive dividends generates no bubble whatever the level of interest rates. However, bubbles of an asset without intrinsic value (Tirole, 1985) may exist at the steady state when interest rates are low. This is the fundamental difference between bubbles of assets with and without dividends.
4. When there is no bubble, the structure of the asset becomes that of the rent introduced by Tirole (1985). Corollary 1 also reminds us Proposition 7 in Tirole (1985) who considers a model with money in the utility function. Bubble formation rests on transactions and speculative demand for money. Dividends (on money) are reinterpreted by Tirole in terms of (marginal) utility, while, in our paper, asset dividends are paid in consumption units. Tirole (1985) shows that positive returns on money rule out the possibility of bubbles; by contrast, in our model, bubbles may appear when dividends tend to zero (see Section 6).

5. Transitional dynamics of capital stocks and asset values

In this section, we provide general results about the equilibrium transition for capital stocks and asset values. According to Lemma 1, the interior equilibrium system is written

$$b_{t+1} = b_t \frac{f'(k_{t+1})}{n} - \xi_{t+1} \text{ and } k_{t+1} = \mathcal{G}_x(k_t, b_t, \xi_t) \tag{18}$$

with

$$b_t > 0 \text{ and } k_{t+1} > 0. \tag{19}$$

(18) is a two-dimensional system with an infinite number of parameters, including the degree of forward altruism x and the sequence of exogenous dividends (ξ_t). Systems of this kind are difficult to handle. Nevertheless, we have obtained equilibrium existence in Lemma 2.

We first look at the set of equilibrium trajectories and, then, we give some asymptotic results. We observe that, for each $b_0 > 0$, there exists a unique sequence $(k_t, b_t)_{t \geq 0}$ satisfying (18). So, given an equilibrium (k_{t+1}, b_t), the asset fundamental value FV_0 at date 0 can be computed through b_0. Hence, we write $FV_0 = FV_0(b_0)$. The initial asset value b_0 affects the size of bubbles $b_0 - FV_0(b_0)$ along the equilibrium transition, and indeterminacy of initial bubble entails in turn the multiplicity of bubbly equilibria. The following lemma is one of the main contributions of the paper.

Lemma 3. *Let Assumptions 1–3 be satisfied.*

1. *The set $\mathcal{B}_0$ of all the values $b_0 > 0$ such that the sequence $(k_{t+1}, b_t)_{t \geq 0}$ determined by (18) is an equilibrium, is an interval.*
2. *The fundamental value function $FV_0(b_0)$ is decreasing in b_0 while the size of bubble $b_0 - FV_0(b_0)$ is strictly increasing.*
3. *There exists at most one bubbleless solution. Moreover, if there are two equilibria with initial asset values $b_{0,1} < b_{0,2}$, then any equilibrium with initial asset value $b_0 \in (b_{0,1}, b_{0,2}]$ is bubbly.*

Proof. See Appendix B.1. □

According to (18), it is easy to see that $k_{t+1} \leq \mathcal{G}_x(k_t, 0, \bar{\xi})$, where $\bar{\xi} \equiv \sup_t \xi_t$.

Assumption 5. There exists a unique $k_{\bar{\xi},x} > 0$ such that $k_{\bar{\xi},x} = \mathcal{G}_x(k_{\bar{\xi},x}, 0, \bar{\xi})$ with $\mathcal{G}_x(k, 0, \bar{\xi}) > k$ if $k < k_{\bar{\xi},x}$ and $\mathcal{G}_x(k, 0, \bar{\xi}) < k$ if $k > k_{\bar{\xi},x}$.

Under this assumption, it is easy to see that $k_t < \max(k_0, k_{\bar{\xi},x})$ for any t. So, (k_t) is uniformly bounded from above. Therefore, Corollary 1 leads to the following result.

Corollary 2. *Under Assumptions 1–5, a bubble exists only if $\sum_{s=1}^{\infty} \xi_s < \infty$. Consequently, if $\xi_t = \xi > 0$ for any t, there does not exist a steady state associated with a bubble in the asset.*

When $\sum_{s=1}^{\infty} \xi_s = \infty$ there is no bubbly equilibrium, and then, according to point (3) of Lemma 3, there exists a unique equilibrium. Thanks to Assumption 5, we need to focus only on the case $\sum_{s=1}^{\infty} \xi_s < \infty$ to look for economies where bubbles may arise.

Assumption 6. For $b > 0$ and $\xi > 0$ small enough, there exists a unique $k_{b,\xi}$ solution to $\mathcal{G}_x(k, b, \xi) = k$.

Denote by k_x^* the solution to $\mathcal{G}_x(k, 0, 0) = k$. Observe that k_x^* is the level of capital stock at the steady state of the economy without financial asset ($b_t = 0$ and $\xi_t = 0$ for any t). Notice also that $\lim_{b,\xi \to 0} k_{b,\xi} = k_x^*$ and $k_{b,\xi}$ is decreasing in b.

Let us present the main result of the section: the global analysis of dynamics of capital stocks and asset values.

Proposition 2. *Let Assumptions 1–6 be satisfied.*

1. *If $f'(k_x^*) > n$, then there exists a unique equilibrium. This unique equilibrium is bubbleless. In addition, if we add that $\lim_{t\to\infty} \xi_t = 0$, then $\lim_{t\to\infty} b_t = 0$.*
2. *If $f'(k_x^*) < n$ and $\xi_0 \geq \xi_1 \geq \cdots \geq \lim_{t\to\infty} \xi_t = 0$. Denote by x_n the solution to $f'(x) = n$. Then, any equilibrium belongs to one of the following three cases.*

[9] Formally, there is T_0 such that $\xi_t = 0$ for any $t \geq T_0$.

(a) $\liminf_{t\to\infty} k_t < x_n$. *In this case, the equilibrium solution is bubbleless and unique.*
(b) $\lim_{t\to\infty} k_t = k_x^*$ *and* $\lim_{t\to\infty} b_t = 0$.
(c) $\lim_{t\to\infty} k_t = x_n$ *and* $\lim_{t\to\infty} b_t = b_n$ *where* b_n *satisfies* $x_n = \mathcal{G}_x(x_n, b_n, 0)$.[10]

Proof. See Appendix B.2. □

Proposition 2 can be viewed as a generalized version of Proposition 1 in Tirole (1985). The novel point is that we work with non-stationary dividends that rise a challenge, while Tirole (1985) considers an asset with zero dividend (he calls it a *bubble*).[11] Another added value is the role of altruism which we will discuss more in detail in Section 6.1.

Let us provide the intuition for the first part of Proposition 2. Recall that the value of bubble is the discounted value of one unit of asset at the infinity

$$\lim_{T\to\infty} Q_T q_T = \lim_{T\to\infty} \frac{1}{a_0} Q_T b_T n^T = \lim_{T\to\infty} \frac{1}{a_0} \frac{n^T}{\Pi_{\tau=1}^T f'(k_\tau)} b_T.$$

This value depends on the asset value, the population growth rate and the interest rates of the economy. Since the asset value is uniformly bounded from above, and interest rate is high (in the sense that $f'(k_x^*) > n$), the value of bubble will be zero. This is true whatever the level of dividends. Considering a particular case where $\xi_t = 0$ for any t and no altruism ($x = 0$), we recover point (a) of Proposition 1 in Tirole (1985). However, in a more general case as ours, along the unique equilibrium, the asymptotic property of capital stocks and asset values may not hold. There is room for fluctuations in the capital stocks if dividends ξ_t fluctuate.[12]

The second case ($f'(k_x^*) < n$) is much more complicated because of the multiple equilibria arising. However, we get also a novel result: if an equilibrium experiences a bubble, then capital stock and asset value must converge. Asset values may converge to zero or to a positive value.

The following result concludes the section and is a direct consequence of Proposition 2 and Corollary 1.

Corollary 3. *Let Assumptions 1–6 be satisfied. The economy experiences a bubble only if* $f'(k_x^*) \leq n$ *and* $\sum_{t\geq 1} \xi_t < \infty$.

6. Examples

In this section, we consider some particular cases and provide more explicit equilibrium analyses. We also provide some new examples of multiple equilibria with and without bubbles.

6.1. Logarithmic utility and Cobb–Douglas technology

We consider the case of a Cobb–Douglas production function $f(k) = Ak^\alpha$ and a logarithmic utility function $U(c, d) = \ln c + \beta \ln d$ with $\beta > 0$. The income sharing between consumption and total saving is given by

$$c_t = \frac{1}{1+\beta}(w_t + g_t) \quad \text{and} \quad s_t + q_t a_t = \frac{\beta}{1+\beta}(w_t + g_t).$$

The equilibrium system is explicitly written

$$nk_1 + b_0 = \frac{\beta}{1+\beta}(w_0 + g_0)$$

$$k_{t+1} = \frac{\alpha A \gamma_x k_t^\alpha + (1-\sigma)\xi_t - \sigma b_t}{n} \quad \forall t \geq 1 \tag{20}$$

$$b_{t+1} = \frac{\alpha A b_t}{n k_{t+1}^{1-\alpha}} - \xi_{t+1} \quad \forall t \geq 0 \tag{21}$$

$$b_t > 0, k_{t+1} > 0, \quad \forall t \geq 0 \tag{22}$$

with the following parameters indexed in the degree of altruism (x):

$$\gamma_x \equiv \frac{\beta}{1+\beta}\frac{1-\alpha+x}{\alpha(1+x)}, \quad \theta_x^* \equiv \frac{\alpha(\gamma_x - 1)}{\sigma} \quad \text{and}$$
$$\sigma \equiv 1 - \frac{\beta}{1+\beta}\frac{x}{1+x}.$$

With our explicit production and utility functions, we compute the reduced functions:

$$\mathcal{G}_x(k, b_t, \xi_t) = \frac{\alpha A \gamma_x k^\alpha + (1-\sigma)\xi_t - \sigma b_t}{n}$$
$$\mathcal{G}_x(k, 0, 0) = \frac{\alpha A \gamma_x}{n} k^\alpha, \quad k_x^* \equiv (\alpha A \gamma_x / n)^{1/(1-\alpha)}.$$

Remark 5.

1. We observe that $\gamma_x = n/f'(k_x^*)$, so condition $f'(k_x^*) < n$ becomes equivalent to $\gamma_x > 1$. Parameter γ_x captures the distortion with respect to the Golden Rule.
2. Under Cobb–Douglas technology and Assumption 4, we see that (k_t) is uniformly bounded from above.

It is easy to check that these specifications satisfy Assumptions 1, 2, 3, 5, 6. Moreover, according to Corollary 2, an equilibrium is bubbly only if $\sum_{t\geq 1} \xi_t < \infty$, the case we will focus on. Consequently, Proposition 2 applies. The following result complements Proposition 2.

Proposition 3. *Assume that* $f(k) = Ak^\alpha$ *and* $U(c, d) = \ln c + \beta \ln d$ *with* $0 < \beta < 1$. *Suppose also that* $\xi_t > 0$ *for any* t *and* $\lim_{t\to\infty} \xi_t = 0$.

1. *If* $\gamma_x < 1$ *(i.e.,* $f'(k_x^*) > n$*), there is a unique equilibrium, which is bubbleless and* $\lim_{t\to\infty} b_t = 0$. *Moreover, if this equilibrium satisfies* $\liminf_{t\geq 0} k_t > 0$*, then the ratio of asset value to production tends to zero:* $\lim_{t\to\infty} b_t/(Ak_t^\alpha) = 0$.
2. *If* $\gamma_x > 1$ *(i.e.,* $f'(k_x^*) < n$*). Let* $(b_t, k_{t+1})_{t\geq 0}$ *be an equilibrium, then there are three cases:*
   (a) $\liminf_{t\to\infty} k_t = 0$.
   (b) *The sequence* (b_t) *converges to* 0*, and* (k_t) *converges to* $k_x^* \equiv (\alpha A \gamma_x / n)^{1/(1-\alpha)}$.
   (c) *The sequence* (b_t) *converges to* $b = n(\gamma_x - 1)x_n$*, and* (k_t) *converges to* $x_n \equiv (\alpha A/n)^{1/(1-\alpha)}$.

Proof. See Appendix C.1. □

Remark 6 (*Comparative Statics*)**.** The limit of capital stock k_x^* in case (2.b) of Proposition 3 increases in the degree of forward altruism (x). The limit of asset value $b = n(\gamma_x - 1)x_n$ in case (2.c) of Proposition 3 increases in x.

These positive effects are intuitive and from the form of forward altruism: bequests are proportional to consumption of old and they improve income, and then saving of young people. The more the savings of the young, the higher the amount at their disposal to buy the financial asset and/or the physical capital.

In Proposition 3, with Cobb–Douglas technology and logarithmic utility function, the dynamical system is more simple and we obtain more analyses with respect to Proposition 2. Precisely, the second part of Proposition 3 does not require the decreasing

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[10] As in Tirole (1985), we do not consider the nongeneric case $f'(k_x^*) = n$ in our paper.

[11] Tirole (1985) also considers another asset that brings stationary dividend (or rent). However, he implicitly assumed that there does not exist bubble in this asset.

[12] See Le Van and Pham (2016) for an analysis in an infinite-horizon setting.

property of dividends sequence and point (2.a) is $\liminf_{t\to\infty} k_t = 0$ instead of $\liminf_{t\to\infty} k_t < x_n$ as in Proposition 2.

Let us explain the idea of the second part of Proposition 3. First, logarithmic utility function implies that the saving rate is constant and Cobb–Douglas technology entails that both income from physical capital $R_t k_t$ and salary w_t are always proportional to the production $f(k_t)$. We then obtain the following key equation (see Appendix C.1 for more details)

$$\frac{b_{t+1}}{Ak_{t+1}^{\alpha}} = \frac{b_t}{Ak_t^{\alpha}} \frac{\alpha}{\alpha\gamma_x + (1-\sigma)\,\xi_t/\left(Ak_t^{\alpha}\right) - \sigma b_t/\left(Ak_t^{\alpha}\right)} - \frac{\xi_{t+1}}{Ak_{t+1}^{\alpha}}.$$

When $\liminf_{t\to\infty} k_t > 0$ and $\lim_{t\to\infty} \xi_t = 0$, the sequence $\xi_t/\left(Ak_t^{\alpha}\right)$ converges to zero. By consequence, in the long run we can obtain the convergence of the ratio of asset value to production $b_t/(Ak_t^{\alpha})$, and hence of b_t and of k_t.

Points (2.b) and (2.c) correspond to part (b) of Proposition 1 in Tirole (1985). It should be noticed that Tirole (1985) does not consider the case where $\liminf_{t\to\infty} k_t$ may be zero. However, this case may be possible. The following example provides an illustration.

Example 1 (*Equilibrium with* $\lim_{t\to\infty} k_t = 0$)**.** Consider the selfish economy ($x = 0$) with production and utility functions: $f(k) = Ak^{\alpha}$ and $U(c, d) = \ln c + \beta \ln d$ with $0 < \beta < 1$.

Let us construct the sequence of dividends (ξ_t) as follows. First, we introduce λ and (x_t) by

$$\lambda \equiv \frac{\alpha^2 + \sqrt{\alpha^4 + 4\alpha^3(1-\alpha)}}{2(1-\alpha)} > \max\left\{1, \ln\left(1 + \frac{2}{\gamma_0}\right)\right\}$$

and $x_t \equiv \max\left\{e^{\lambda^t}, 1 + 2/\gamma_0\right\}$

where $\gamma_0 \equiv \dfrac{1-\alpha}{\alpha}\dfrac{\beta}{1+\beta}$.

Second, we define a sequence $(\bar{b}_t, \bar{k}_{t+1}, \xi_t)$ by

$$\bar{b}_t = \alpha A\gamma_0 \bar{k}_t^{\alpha} - n\bar{k}_{t+1} \text{ and } \bar{k}_{t+1} = \frac{\alpha A\bar{k}_t^{\alpha}}{nx_t}$$

$$\xi_{t+1} \equiv \frac{\alpha A\bar{b}_t}{n\bar{k}_{t+1}^{1-\alpha}} - \bar{b}_{t+1}. \tag{23}$$

With this setup, $\lim_{t\to\infty} \xi_t = 0$.

In this economy with above fundamentals, the sequence $(b_t, k_{t+1})_{t\geq 0}$, determined by $(b_t, k_{t+1}) = (\bar{b}_t, \bar{k}_{t+1})$ for any $t \geq 0$, is the unique equilibrium, and it satisfies $\lim_{t\to\infty} k_t = \lim_{t\to\infty} b_t = 0$.

Proof. See Appendix C.2. □

Example 1 indicates that there is an economy with Cobb–Douglas technology, in which there exists an equilibrium with $\lim_{t\to\infty} k_t = \lim_{t\to\infty} b_t = 0$. In this example, the sequence of dividends (ξ_t) is strictly positive but converges to zero. However, dividends in first days are very high comparing to capital stock. This causes the concentration of savings in asset instead of physical capital, which implies the same situation for the next day, and so on to infinity. The aggregate capital hence converges to zero. However, we can verify the ratio of dividend on capital converges to infinity, i.e., $\lim_{t\to\infty} \xi_t/\bar{k}_t = \infty$ (for details, see Appendix C.2). One can prove, by using the same argument in the proof of point (2.a) in Proposition 2 and noticing that $\lim_{t\to\infty} k_t = 0$ in Example 1, that this equilibrium is the unique equilibrium of the economy.

In the economy in Example 1, if we exclude the positive dividends, then we recover the standard model in which $\lim_{t\to\infty} k_t = x_n \equiv (\alpha A/n)^{1/(1-\alpha)}$ for any $k_0 > 0$. So, Example 1 suggests an interesting property: the presence of financial asset having dividends (ξ_t) may create a collapsing equilibrium (in the sense that $\lim_{t\to\infty} k_t = \lim_{t\to\infty} b_t = 0$). This result recalls us to a well-known "resources curse", though the situation in our article is not exactly the same discussed in the literature.

We now illustrate and complement point 2 of Proposition 3 by providing an example where $\liminf_{t\to\infty} k_t > 0$ and there are multiple bubbly equilibria. Following Corollary 2, we will choose dividends decrease geometrically.

Example 2 (*Continuum of Bubbly Equilibria with Forward Altruism*)**.** Let $\xi_t \equiv \xi/n^t$ with $n > \gamma_x > 1$ and

$$k_m \equiv \min\{k_0, x_n\} \leq x_n \equiv \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}} < \bar{k} \equiv \left(\frac{\alpha A\gamma_x}{n}\right)^{\frac{1}{1-\alpha}}$$
$$\leq k_M \equiv \max\left\{k_0, \bar{k}\right\}$$
$$\xi \in (0, \bar{\xi})$$

where $\bar{\xi} > 0$ is solution to

$$\frac{\alpha}{\alpha\gamma_x + (1-\sigma)\,\xi/(Ak_M^{\alpha})} = \frac{1}{n} + \frac{\xi}{\theta_x^* Ak_m^{\alpha}}.$$

Then, any sequence $(b_t, k_{t+1})_{t\geq 0}$ determined by the system (20)–(21) and b_0 such that $\theta_x^* Ak_0^{\alpha}/n < b_0 < \theta_x^* Ak_0^{\alpha}$, is an equilibrium. By consequence, according to Lemma 3, there are continuum equilibria with bubble. Moreover, $k_t \geq k_m > 0$ for any t.

Proof. See Appendix C.3. □

In Example 2, the dividends are not so high comparing to the physical capitals, so the capital sequence is bounded away from zero. By Proposition 3, it converges either to k_x^* with $f'(k_x^*) < n$, either to x_n with $f'(x_n) = n$. In Example 2, there exist a continuum of equilibria and hence a continuum of bubbly equilibria. For every equilibrium, except the one having the positive bubble component, the asset value b_t converges to zero. These results are consistent with the analysis in the case where $\xi_t = 0$ for any t as in Tirole (1985).

Remark 7. Bubbles arise in an OLG model à la Diamond (Tirole, 1985). However, under positive bequests, an arbitrarily small degree of altruismà la Barro (1974) immediately kills the bubble in models à la Diamond (Bosi et al., 2016).

In our paper, forward altruism is based on constraints instead of utility. In this case, bubbles may arise in OLG models with altruism. The reason is that bequests from old to young are proportional to consumption of old. The old people finance these bequests and partly purchase the bubble when young.

6.1.1. Explicit solution in the case of pure bubble

In this section, we consider the dynamics of pure bubbles à la Tirole (1985) by setting $\xi_t = 0$ for any t. In this case, the value of bubble equals the asset value. We provide the explicit trajectories of both capital stocks and asset values.

The equilibrium system is written

$$nk_1 + b_0 = \frac{\beta}{1+\beta}(w_0 + g_0) \tag{24}$$

$$nk_{t+1} + \sigma b_t = \gamma_x \alpha Ak_t^{\alpha} \forall t \geq 1 \tag{25}$$

$$nb_{t+1} = \alpha Ak_{t+1}^{\alpha-1} b_t \tag{26}$$

with $k_{t+1} > 0$, $b_t \geq 0$, where

$$\sigma \equiv 1 - \frac{\beta}{1+\beta}\frac{x}{1+x} \in (0, 1],$$

$$\gamma_x \equiv \frac{\beta}{1+\beta}\frac{1-\alpha+x}{\alpha(1+x)} = \frac{n}{f'\left(k_x^*\right)}.$$

Here, k_x^* is the capital intensity in the bubbleless steady state, that is the steady state solution of (25) with $b = 0$:

$$k_x^* = \rho_{\gamma_x}^{1/(1-\alpha)} \tag{27}$$

with $\rho_{\gamma_x} \equiv \gamma_x \alpha A/n$. We eventually introduce the bubble critical value:

$$\bar{b}_x \equiv (w_0 + g_0)\frac{\beta}{1+\beta}\frac{\gamma_x - 1}{\gamma_x - 1 + \sigma} = (w_0 + g_0)\left[1 - \frac{1 + x + \alpha\beta}{(1+x)(1-\alpha)(1+\beta)}\right] \quad (28)$$

which is positive if $\gamma_x > 1$.

These elements allow us to introduce the main result of this section.

Proposition 4. *Assume that $f(k) = Ak^\alpha$, $U(c, d) = \ln c + \beta \ln d$ with $0 < \beta < 1$, and $\xi_t = 0$ for any t.*

1. *If $\gamma_x \le 1$ (i.e. $f'(k_x^*) \ge n$), the equilibrium is unique and bubbleless and the equilibrium sequence of capital intensities is given by*

$$k_t = \rho_{\gamma_x}^{\frac{1-\alpha^{t-1}}{1-\alpha}} k_1^{\alpha^{t-1}} \quad \forall t \ge 2, \quad k_1 = \frac{\beta}{n(1+\beta)}(w_0 + g_0). \quad (29)$$

Moreover, $\lim_{t\to\infty} k_t = k_x^*$, where k_x^* is given by (27).

2. *If $\gamma_x > 1$ (i.e. $f'(k_x^*) < n$), the equilibrium is indeterminate. The set of equilibria $(k_{t+1}, b_t)_{t\ge 0}$ is defined by (25), (26), and $b_0 \in [0, \bar{b}_x]$. Moreover,*
   (a) *(bubbleless equilibrium) If $b_0 = 0$, and, thus, $b_t = 0$ forever. The sequence (k_t) is given by (29).*
   (b) *(bubbly equilibrium) If $b_0 > 0$, then $b_t > 0$ for any t. When $b_0 < \bar{b}_x$, we have $\lim_{t\to\infty} b_t = 0$ and $\lim_{t\to\infty} k_t = k_x^*$. When $b_0 = \bar{b}_x$, we have $\lim_{t\to\infty} b_t > 0$. We also have*

$$b_t = \frac{\gamma_x - 1}{\sigma} n k_{t+1} \forall t \ge 0 \quad (30)$$

$$k_t = \rho_1^{\frac{1-\alpha^{t-1}}{1-\alpha}} k_1^{\alpha^{t-1}} \quad \forall t \ge 2, \quad k_1 = \frac{\alpha(w_0 + g_0)}{n(1-\alpha)}\left(1 - \frac{\beta}{1+\beta}\frac{x}{1+x}\right) \quad (31)$$

and $\rho_1 \equiv \alpha A/n$. Moreover,

$$\lim_{t\to\infty} k_t = \rho_1^{1/(1-\alpha)} < k_x^* \text{ and } b_x \equiv \lim_{t\to\infty} b_t = n\frac{\gamma_x - 1}{\sigma}\rho_1^{1/(1-\alpha)} > 0. \quad (32)$$

Proof. See Appendix C.4. □

Definition 3. $\bar{b}_x$ is the (upper) size of bubbly asset value at initial date with forward altruism (in the case $\gamma_x > 1$).

The value $\rho_1^{1/(1-\alpha)}$ corresponds to the value x_n determined by $f'(x_n) = n$ and introduced in Proposition 2.

Proposition 4 illustrates and complements Proposition 2 in the case $\xi_t = 0$ for any t. It is instructive to compare these two propositions. Proposition 4 supplies a number of new results: explicit equilibrium sequences, a proof of global convergence, a necessary and sufficient condition for bubble existence as well as for equilibrium indeterminacy. All these issues remain unaddressed in theoretical papers.[13]

Another added value of this section is that we can compute explicitly $\bar{b}_x$, the maximum feasible bubble at the initial date, in terms of fundamental parameters. Indeed, recall that Tirole (1985) only proves the existence of such the maximum level. However, under specifications in Proposition 4, this level can be computed by (28).

The explicit form also allows us to analyze the impact of some relevant parameter (impatience and altruism) on equilibrium trajectories.

Comparative statics.

1. (existence of bubble). Condition $\gamma_x \equiv n/f'(k_x^*) > 1$ (i.e. low interest rates or capital overaccumulation) is equivalent to
$$\frac{\alpha(1+x)}{1-\alpha+x} < \frac{\beta}{1+\beta}. \quad (33)$$
The left-hand side of (33) decreases with x. Thus, forward altruism promotes the emergence of bubbles.
2. Both the limits k_x^* and b_x increase in x. The intuition is similar to that in Remark 6.
3. (maximum value $\bar{b}_x$). Let us compute the effects of initial capital, patience and altruism on the maximum level of asset value. According to (28), we have
$$\frac{\partial \bar{b}_x}{\partial k_0}, \frac{\partial \bar{b}_x}{\partial \beta}, \frac{\partial \bar{b}_x}{\partial x} > 0.$$
4. (equilibrium transition). Consider the case of low interest rates (i.e., $f'(k_x^*) < n$ or $\gamma_x > 1$). Look at the asymptotically bubbly equilibrium (i.e. $b_0 = \bar{b}_x$). We see that $b_0 = \bar{b}_x$ increases in x, so k_1 determined by (24) decreases in x. Since $k_{t+1} = \rho_1 k_t$ for any t, we see that k_t decreases in x for any t. Hence, R_t increases in x for any t. By using the induction argument and the fact that $b_t = R_t b_{t-1}/n$ for any $t \ge 0$, we obtain that b_t increases in x for any t. So, along the asymptotically bubbly equilibrium, asset value[14] b_t increases but capital stock k_t decreases in the forward altruism degree.

6.2. Logarithmic utility and linear technology

We consider the case of a linear production function $F(K, L) = RK + wL$ and logarithmic utility function $U(c_t, d_{t+1}) = \ln c_t + \beta \ln d_{t+1}$ with $0 < \beta < 1$. In this case, we have

$$c_t = \frac{1}{1+\beta}(w_t + g_t) \text{ and } s_t + q_t a_t = \frac{\beta}{1+\beta}(w_t + g_t)$$
$$g_t = \frac{x}{1+x}\left(k_t f'(k_t) + b_t + \xi_t\right) \ \forall t \ge 1$$

with g_0 given. The equilibrium system becomes

$$nk_{t+1} + \left(1 - \frac{\beta}{1+\beta}\frac{x}{1+x}\right)b_t = R\frac{\beta}{1+\beta}\frac{x}{1+x}k_t + \frac{\beta}{1+\beta} \times \left(w + \frac{x}{1+x}\xi_t\right) \ \forall t \ge 1 \quad (34)$$

$$b_{t+1} + \xi_{t+1} = \frac{R}{n}b_t \quad (35)$$

with $k_t > 0$ and $b_t > 0$. Notice that, at the initial date, $nk_1 + b_0 = \frac{\beta}{1+\beta}(w + g_0)$.

We compute the fundamental value of financial asset:

$$FV_0 = \sum_{t=1}^{\infty}\frac{\delta_t}{R^t} = \sum_{t=1}^{\infty}\frac{n^t \xi_t}{R^t a_0}.$$

Solving recursively no-arbitrage condition in (35) yields

$$b_t = \frac{R^t}{n^t}\left(b_0 - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right). \quad (36)$$

We now present the main result of this section, which characterizes all equilibria.

[13] Bosi and Seegmuller (2013) show the local indeterminacy of real bubbles (rational exuberance). We focus instead on global indeterminacy of real bubbles.

[14] When dividends are zero, asset value and bubble coincide.

Proposition 5. *Assume that $F(K,L) = RK + wL$ and $U(c,d) = \ln c + \beta \ln d$ with $0 < \beta < 1$. At equilibrium, we have*

$$nk_{t+1} + b_t = D^t \frac{\beta}{1+\beta}(w+g_0) + \frac{\beta w}{1+\beta}\frac{1-D^t}{1-D} \text{ where}$$
$$D \equiv \frac{R}{n}\frac{\beta}{1+\beta}\frac{x}{1+x}. \quad (37)$$

Hence,

1. *$R > n$. There is no bubbly equilibrium.*
2. *$R \leq n$. Assume that*

$$D^t \frac{\beta}{1+\beta}(w+g_0) + \frac{\beta w}{1+\beta}\frac{1-D^t}{1-D} - \frac{R^t}{n^t} \times \left[\frac{\beta}{1+\beta}(w+g_0) - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right] > 0 \ \forall t \geq 1.$$

Then, the set of interior equilibria $(k_{t+1}, b_t)_{t\geq 0}$ is determined by conditions (36), (37) and

$$b_0 \in \left[a_0 FV_0, \frac{\beta}{1+\beta}(w+g_0)\right).$$

If $b_0 > a_0 FV_0$, then the equilibrium is bubbly; moreover, in this case, $\lim_{t\to\infty} b_t > 0$ if and only if $R = n$.

Proof. See Appendix C.5. □

Condition $R > n$ (resp. $R < n$) corresponds to the case of high (resp. low) interest rate in Proposition 2. Thanks to specifications in Proposition 5 we can compute and get a complete characterization of interior equilibrium paths.

More economic implications of Proposition 5 will be presented in Section 7.

Remark 8 (*No Interior Equilibrium*)**.** According to (37), we see that

$$D^t \frac{\beta}{1+\beta}(w+g_0) + \frac{\beta w}{1+\beta}\frac{1-D^t}{1-D} > b_t = \frac{R^t}{n^t}\left(b_0 - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right) \geq \sum_{s=t+1}^{\infty}\frac{n^{s-t}}{R^{s-t}}\xi_s.$$

Hence, there is no interior equilibrium if

$$D^t \frac{\beta}{1+\beta}(w+g_0) + \frac{\beta w}{1+\beta}\frac{1-D^t}{1-D} < \sum_{s=t+1}^{\infty}\frac{n^{s-t}}{R^{s-t}}\xi_s \ \forall t.$$

This happens when the productivity R is low. The intuition is that, when the productivity is low, households tend to invest in financial asset rather than in physical capital. Therefore, the capital stock k_t may be zero.

7. Bubble, asset price and interest rate revisited

7.1. Does the existence of bubbles really require low interest rates and low dividends?

The seminal article by Tirole (1985) proves that pure bubbles may arise if the interest rate at the steady state of the economy without financial asset is below the population growth rate. As shown above, this well-known result still holds for an asset bringing non-stationary dividends in an altruistic economy. Both findings are based on the boundedness of both production (per capital) $f(k_t)$ and dividend (per capital) ξ_t.

In this section, we revisit this result. Precisely, we consider an economy where the output may grow, and we wonder whether existence of bubble still requires low interest rates and low dividends conditions.

For the sake of simplicity, we reconsider the setup in Proposition 5 but with a non-stationary linear technology: $F_t(K,L) = RK + w_t L$, where $R, w_t > 0$ are exogenous. The equilibrium system becomes

$$nk_{t+1} + \left(1 - \frac{\beta}{1+\beta}\frac{x}{1+x}\right)b_t = R\frac{\beta}{1+\beta}\frac{x}{1+x}k_t + \frac{\beta}{1+\beta} \times \left(w_t + \frac{x}{1+x}\xi_t\right) \ \forall t \geq 1 \quad (38)$$
$$b_{t+1} + \xi_{t+1} = \frac{R}{n}b_t.$$

We have, as (36),

$$b_t = \frac{R^t}{n^t}\left(b_0 - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right). \quad (39)$$

Let us provide an example where (1) bubbles appear, (2) $R > n$ and (3) ξ_t may be unbounded. To do so, we choose (w_t, ξ_t) and b_0 such that

$$\frac{\beta}{1+\beta}w_t > \left(1 - \frac{\beta}{1+\beta}\frac{x}{1+x}\right)\frac{R^t}{n^t}\left(b_0 - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right) \quad (40)$$
$$b_0 \in \left[\sum_{s=1}^{\infty}\frac{n^s}{R^s}\xi_s, \frac{\beta}{1+\beta}(w+g_0)\right]. \quad (41)$$

Then the sequence (k_{t+1}, b_t) defined by (38), (39), (40) and (41) is an interior equilibrium; moreover bubbles appear if and only if $b_0 > \sum_{s=1}^{\infty}\frac{n^s}{R^s}\xi_s$.

The economic intuition of our counterexample is the following. When the productivity on the capital-free side of production w_t grows, the labor income of households increases. If such a productivity is high enough, salary w_t may growth faster than $(R/n)^t$, and in this case it would be high enough to cover the value of asset with bubbles that agents may buy. By consequence, even when the interest rate is high (i.e., $R > n$), there may be a bubble.

Remark 9. Weil (1990) considers on OLG model where consumers receive exogenous dividends. However, Weil (1990) (p. 1469) assumes that dividends become zero from some date on to allow for the possibility of bubble. However, the above example shows that a bubbly equilibrium is possible even if dividends are positive at any date and may tend to infinity.

7.2. Bubbles and monotonicity of asset prices

By definition, an asset bubble appears when the asset price is strictly higher than the asset fundamental value. Some authors are interested in checking whether a causal link holds between the existence of asset bubble and the rise of asset price. Weil (1990) explains why along a bubbly equilibrium the asset prices may decrease. In Proposition 5, under a linear technology, the asset price at date 0 is given by

$$q_t = \frac{b_t}{a_t} = \frac{b_t n^t}{a_0} = \frac{R^t}{a_0}\left(b_0 - a_0 FV_0 + \sum_{s=t+1}^{\infty}\frac{n^s \xi_s}{R^s}\right).$$

We see that the asset price q_t may increase or decrease or even fluctuate (in time) along a bubbly equilibrium. In other words, there is no causal relationship between the existence of bubbles and monotonicity of asset prices. The reader is referred to Bosi et al. (forthcoming) for a similar finding but in an infinite-horizon general equilibrium with heterogeneous agents and financial frictions.

8. Conclusion

We have introduced two additional ingredients in an OLG model à la Diamond (1965): an asset bringing positive dividends and a kind of descendent altruism. We have shown that bubbles are ruled out if the sum (over time) of ratios of dividend to output is finite. When outputs are bounded from above, the economy experiences a bubble only when (1) interest rates remain below the population growth factor and (2) the sum (over time) of dividends is finite. Some examples of multiple bubbly equilibria have been provided. However, when outputs are not bounded, bubbles may appear even if the interest rates are greater than the population growth rates or even if dividends do not converge to zero (or even if they tend to infinity).

In standard framework, the forward altruism promotes pure bubble à la Tirole (1985) and has a positive impact on asset values but a negative impact on the capital stocks along the transition sequence of an asymptotically bubbly equilibrium.

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Appendix A. Proofs of Section 3

A.1. Proof of Lemma 1

Since $\frac{b_{t-1}f'(k_t)}{n} = b_t+\xi_t$, Eq. (12) is equivalent to $H_{k_t,b_t,\xi_t}(k_{t+1}) = 0$ where

$$H_{k_t,b_t,\xi_t}(k) \equiv nk + b_t - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_tf'(k_t) + \frac{x}{1+x}(b_t+\xi_t), f'(k)\Big) = 0 \quad \text{for } t \geq 1.$$

The saving function $\mathcal{S}_x$ is increasing in $f'(k)$ because $cu'(c)$ is increasing in x. So, the function $H_{k_t,b_t,\xi_t}(k)$ is increasing in k. Since we assume that (11)–(14) has a solution, we define $\mathcal{G}_x(k_t, b_t, \xi_t)$ the solution of equation $H_{k_t,b_t,\xi_t}(k) = 0$, and hence $k_{t+1} = \mathcal{G}_x(k_t, b_t, \xi_t)$. It is easy to see that $\mathcal{G}_x$ is continuously differentiable.

We can see that $\frac{\partial \mathcal{S}_x}{\partial(w+g)}(w+g, R) \in (0, 1)$.[15] Hence, $H_{k_t,b_t,\xi_t}(k)$ increases if b_t increases. By consequence, $\mathcal{G}_x(k_t, b_t, \xi_t)$ is decreasing in b_t. It is easy to see that $\mathcal{G}_x(k_t, b_t, \xi_t)$ is increasing in both k_t and ξ_t.

According to (10), we have

$$nk_{t+1} + \frac{b_t}{1+x} \leq f(k_t) - \frac{k_tf'(k_t)}{1+x} + \frac{x\xi_t}{1+x}. \tag{A.1}$$

As a result, we get $\mathcal{G}_x(k, +\infty, \xi) = -\infty$.

The property $\mathcal{G}_x(k, 0, \xi) > 0$ for any $k > 0$ any $\xi > 0$ is easily proved by using the definition of $\mathcal{G}_x$.

[15] Indeed, given $W > 0$ and $R > 0$, the function $\mathcal{S}_x$ is defined by $u'(W - \mathcal{S}_x) = \frac{\beta R}{1+x}u'\Big(\frac{R}{1+x}\mathcal{S}_x\Big)$. Taking the derivative of both sides, we get

$$u''(W-\mathcal{S}_x)\Big(1 - \frac{\partial \mathcal{S}_x}{\partial W}\Big) = \frac{\beta R^2}{(1+x)^2}u''\Big(\frac{R}{1+x}\mathcal{S}_x\Big)\frac{\partial \mathcal{S}_x}{\partial W}.$$

Since $\frac{\partial \mathcal{S}_x}{\partial W} > 0$, we have $\frac{\partial \mathcal{S}_x}{\partial W} < 1$.

A.2. Proof of Lemma 2

We prepare our proof by intermediate steps. First, we prove the following claim.

Claim 1. *Let a positive sequence (ξ_t) be given. Consider a date $t \geq 1$. Given $b_{t+1} \geq 0$, $k_t > 0$, there exist b_t and k_{t+1} such that*

$$nk_{t+1} + b_t - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_tf'(k_t) + \frac{x}{1+x}(b_t+\xi_t), f'(k_{t+1})\Big) = 0 \quad \text{for } t \geq 1$$

$$b_{t+1} = \frac{b_tf'(k_{t+1})}{n} - \xi_{t+1}$$

$$b_t > 0, \quad k_{t+1} > 0.$$

Moreover, k_{t+1} and b_t are continuously increasing in k_t.

Proof of Claim 1. It is sufficient to prove that there is $k_{t+1} > 0$ such that

$$nk_{t+1} + \frac{n\,(b_{t+1}+\xi_{t+1})}{f'\,(k_{t+1})} - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_tf'(k_t) + \frac{x}{1+x}\Big(\frac{n}{f'\,(k_{t+1})} \times (b_{t+1}+\xi_{t+1}) + \xi_t\Big), f'(k_{t+1})\Big) = 0 \quad \text{for } t \geq 1. \tag{A.2}$$

First, as in the proof of Lemma 1, we see that $\frac{\partial \mathcal{S}_x}{\partial a_1}(a_1, a_2) \in (0, 1)$. So, combining this with the fact that the saving function is increasing in $f'(k_{t+1})$, we can verify that the left-hand side of the above equation is an increasing function on k_{t+1}. Moreover, since $f'(0) = \infty$, the left-hand side is negative when k_{t+1} is small enough and positive when k_{t+1} is high enough. Therefore, Eq. (A.2) has a unique solution $k_{t+1} > 0$. For such $k_{t+1} > 0$, we determine b_t by

$$b_t = \frac{n}{f'\,(k_{t+1})}\,(b_{t+1}+\xi_{t+1})\,.$$

It is easy to see that the function $f(k_t) - \frac{1}{1+x}k_tf'(k_t)$ is increasing in k_t. So, combining this with Eq. (A.2) and property of the saving function, we see that k_{t+1} and b_t are continuously increasing in k_t. □

By using the same argument in the proof of Claim 1, we also get the following result.

Claim 2. *Let a positive sequence $(\xi_t)_{t\geq 0}$, and $b_1 \geq 0$, $k_0 > 0$, $g_0 > 0$ be given. There exist $b_0 > 0$ and $k_1 > 0$ such that*

$$nk_1 + b_0 - \mathcal{S}_x\Big(w_0 + g_0, f'(k_1)\Big) = 0, \quad b_1 = \frac{b_0f'(k_1)}{n} - \xi_1.$$

Moreover, k_1 and b_0 are continuously increasing in k_0.

We now prove the following result.

Claim 3. *Let a positive sequence (ξ_t) be given. Consider an integer $T \geq 1$. Given $k_1 > 0$, $b_{T+1} = 0$, there exists $(k_{t+1}, b_t)_{t=1}^T$ such that, for any $1 \leq t \leq T$,*

$$nk_{t+1} + b_t - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_tf'(k_t) + \frac{x}{1+x}(b_t+\xi_t), f'(k_{t+1})\Big) = 0$$

$$b_{t+1} = \frac{b_tf'(k_{t+1})}{n} - \xi_{t+1}$$

$$b_t > 0, \quad k_{t+1} > 0.$$

and k_2, b_1 are continuously increasing in k_1.

Proof of Claim 3. We prove by using the induction argument (with respect to T). According to Claim 1, Claim 3 holds for $T = 1$.

Suppose that Claim 3 holds until T. Let us prove it for $T + 1$.

Let $k_2 > 0$, $b_{T+1} = 0$. Since Claim 3 holds for the integer T, there exists $(k_{t+1}, b_t)_{t=2}^{T+1}$ such that, for any $t = 2, \ldots, T+1$,

$$nk_{t+1} + b_t - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_t f'(k_t) + \frac{x}{1+x}(b_t + \xi_t), f'(k_{t+1})\Big) = 0$$

$$b_{t+1} = \frac{b_t f'(k_{t+1})}{n} - \xi_{t+1}$$

$$b_t > 0, \quad k_{t+1} > 0$$

and b_2 is continuously increasing in k_2.

We now have to prove that there exist $k_2 > 0$ and $b_1 > 0$ such that

$$nk_2 + b_1 - \mathcal{S}_x\Big(f(k_1) - \frac{1}{1+x}k_1 f'(k_1) + \frac{x}{1+x}(b_1 + \xi_1), f'(k_2)\Big) = 0 \quad \forall t \geq 0$$

$$b_2 = \frac{b_1 f'(k_2)}{n} - \xi_2 \quad \forall t \geq 0$$

and k_2, b_1 are continuously increasing in k_1. Note that b_2 depends on k_2.

We can prove this by using the argument in the proof of Claim 1 and the property that b_2 is increasing in k_2.

Note that k_2, b_1 are continuously increasing k_1. $\square$

Claim 4 (T-truncated Equilibrium System). *Let a positive sequence (ξ_t) be given. Consider an integer $T \geq 0$. Given $k_0 > 0, g_0 > 0$, $b_{T+1} = 0$, there exists $(k_{t+1}, b_t)_{t=0}^{T}$ such that*

$$nk_1 + b_0 - \mathcal{S}_x\Big(w_0 + g_0, f'(k_1)\Big) = 0$$

$$nk_{t+1} + b_t - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_t f'(k_t) + \frac{x}{1+x}(b_t + \xi_t), f'(k_{t+1})\Big) = 0$$

$$b_{t+1} = \frac{b_t f'(k_{t+1})}{n} - \xi_{t+1}$$

$$b_t > 0, \quad k_{t+1} > 0$$

for any $1 \leq t \leq T$, and k_1, b_0 are continuously increasing in k_0, g_0.

Proof. According to Claim 2, Claim 4 holds for $T = 0$.

Suppose that Claim 4 holds. Let us prove it for $T + 1$.

Let $k_1 > 0$, $b_{T+1} = 0$. Since Claim 3 holds, there exists $(k_{t+1}, b_t)_{t=1}^{T+1}$ such that, for any $t = 1, \ldots, T+1$,

$$nk_{t+1} + b_t - \mathcal{S}_x\Big(f(k_t) - \frac{1}{1+x}k_t f'(k_t) + \frac{x}{1+x}(b_t + \xi_t), f'(k_{t+1})\Big) = 0$$

$$b_{t+1} = \frac{b_t f'(k_{t+1})}{n} - \xi_{t+1}$$

$$b_t > 0, \quad k_{t+1} > 0$$

and b_1, k_2 are continuously increasing in k_1.

We now have to prove that there exist $k_1 > 0$, $b_0 > 0$ such that

$$nk_1 + b_0 - \mathcal{S}_x\Big(w_0 + g_0, f'(k_1)\Big) = 0$$

$$b_1 = \frac{b_0 f'(k_1)}{n} - \xi_1$$

and b_0, k_1 are continuously increasing in k_0. Notice that b_1 depends on k_1. We can prove this by using the argument in the proof of Claim 1 and the property that b_1 is increasing in k_1. $\square$

We now come back to the proof of Lemma 2. According to previous claims, the T-truncated equilibrium system defined in Claim 4 has a solution $(b_t^T, k_{t+1}^T)_{t\leq T}$.

Let now T tend to infinity: there exists a sub-sequence (t_n) such that $\lim_{n\to\infty}\left(b_t^{t_n}, k_{t+1}^{t_n}\right) = (b_t, k_{t+1})$ for any t. It is easy to see that $(b_t, k_{t+1})_{t\geq 0}$ is a solution to (18).

A.3. Proof of Corollary 1

Condition (10) implies that $b_t \leq (1+x)f(k_t) + x\xi_t$. By combining this with point (iii) of Proposition 1, the existence of bubble implies that

$$\sum_{t=1}^{\infty} \frac{\xi_t}{(1+x)f(k_t) + x\xi_t} < \infty.$$

Since $x > 0$, we have $\lim_{t\to\infty}\xi_t/f(k_t) = 0$. So, there is t_0 such that $(1+x)f(k_t) + x\xi_t < (2+x)f(k_t)$ for any $t \geq t_0$. As a result, we obtain

$$\sum_{s=t_0}^{\infty} \frac{\xi_s}{f(k_s)} < (2+x)\sum_{s=t_0}^{\infty} \frac{\xi_t}{(1+x)f(k_t) + x\xi_t} < \infty.$$

Appendix B. Proofs of Section 5

B.1. Proof of Lemma 3

(1) Consider the two solutions $b_0^1 \leq b_0^2$ with (b_t^1, k_{t+1}^1) and (b_t^2, k_{t+1}^2) two corresponding sequences of asset values and capital stocks. Suppose that $b_0^1 \leq b_0 \leq b_0^2$. Consider the sequence (b_t, k_{t+1}) generated (18). By induction, it is easy to prove that for any $t \geq 0$, we have $k_{t+1}^1 \geq k_{t+1} \geq k_{t+1}^2$ and $b_t^1 \leq b_t \leq b_t^2$. Hence, the sequence (b_t, k_{t+1}) is also a solution of the dynamic system.

(2) Take b_0^1 and b_0^2 as at point (1). For any t, we have $k_{t+1}^1 \geq k_{t+1}^2$ and $b_t^1 \leq b_t^2$, and, therefore, $f'(k_{t+1}^1) \leq f'(k_{t+1}^2)$. Hence, $FV(b_0^1) \geq FV(b_0^2)$ and, if $b_0^1 < b_0^2$, we have $b_0^1 - FV(b_0^1) < b_0^2 - FV(b_0^2)$. The function $b_0 - FV(b_0)$ is strictly increasing.

(3) Since, for any solution, we have $b_0 - FV(b_0) \geq 0$, point (3) is a direct consequence of point (2).

B.2. Proof of Proposition 2

We prepare our proof by the following result.

Lemma 4. *Let Assumptions 1–6 be satisfied. Suppose also that $f'(k_x^*) > n$ and $\lim_{t\to\infty}\xi_t = 0$. Then, $\lim_{t\to\infty}b_t = 0$.*

Proof of Lemma 4. Fix $\bar{x} > k_x^*$ such that $f'(\bar{x}) > n$.

Let $\xi > 0$. There exists $T(\xi)$ such that $\xi_t \leq \xi$ for any $t \geq T(\xi)$. Hence, $k_{T+t} \leq \mathcal{G}_x^t(k_T, 0, \xi)$. Moreover, we see that $\lim_{t\to\infty}\mathcal{G}_x^t(k_T, 0, \xi) = k_{0,\xi}$ for any $k_T > 0$. Hence, we obtain $\limsup_{t\to\infty}k_t \leq k_{0,\xi}$ for any ξ.

Let ξ converge to 0, we have that $k_{0,\xi}$ converges to k_x^*, and hence $\limsup_{t\to\infty}k_t \leq k_x^* < \bar{x}$. So, there exists T high enough such that $k_t \leq \bar{x}$ for any $t \geq T$.

Assume that $\limsup_t b_t > 0$. Let $\epsilon > 0$ satisfy $(\limsup_t b_t)\left[f'(\bar{x})/n - 1\right] - \epsilon > 0$. Then, there exists $T_0 > T$ high enough such that $k_t \leq \bar{x}$ and $\xi_t < \epsilon$ for any $t \geq T_0$. Thus, we have

$$b_T\left[\frac{f'(k_{T+1})}{n} - 1\right] - \xi_{T+1} \geq b_T\left[\frac{f'(\bar{x})}{n} - 1\right] - \epsilon > 0.$$

Therefore, $b_{T+1} - b_T = b_T\left[\frac{f'(k_{T+1})}{n} - 1\right] - \xi_{T+1} > 0$, and hence

$$b_{T+2} - b_{T+1} = b_{T+1}\left[\frac{f'(k_{T+2})}{n} - 1\right] - \xi_{T+2}$$
$$\geq b_{T+1}\left[\frac{f'(\bar{x})}{n} - 1\right] - \epsilon > b_T\left[\frac{f'(\bar{x})}{n} - 1\right] - \epsilon > 0.$$

So, the sequence $(b_t)_{t\geq T}$ is increasing and converges to $\bar{b} < +\infty$, since from (10), $(b_t)_t$ is uniformly bounded from above. Therefore, from the no-arbitrage condition (7), k_{t+1} converges to x_n with $f'(x_n) = n$. This leads to a contradiction since, for all $t \geq T$, $f'(k_t) \geq f'(\hat{x}) > n$. We have proved that b_t converges to 0. □

We now come back to the proof of Proposition 2.

Part 1. Assume that $f'(k^*_x) > n$.

Case 1: $\limsup_{t\to\infty}\xi_t > 0$, according to Corollary 1, every equilibrium is bubbleless.

Case 2: $\lim_{t\to\infty}\xi_t = 0$. For any $\xi > 0$, there exists T such that $\xi_{T+t} < \xi$ for any $t \geq 0$. Hence, $k_{T+t} \leq \mathcal{G}^t_x(k_T, 0, \xi)$, which implies $\limsup_{t\to\infty}k_t \leq k_{0,\xi}$. Let ξ converge to 0, we have that $k_{0,\xi}$ converges to k^*_x, and hence $\limsup_{t\to\infty}k_t \leq k^*_x$.

According to Lemma 4, we have $\lim_{t\to\infty}b_t = 0$. By combining this with condition $\limsup_{t\to\infty}k_t \leq k^*_x$, it is easy to prove that $\limsup_{T\to\infty}\frac{n^T}{\prod_{t=1}^T f'(k_t)}b_T = 0$. So, there is no bubble.

Part 2. Consider now the case $f'(k^*_x) < n$.

CASE 1: Consider the case $\liminf_{t\to\infty}k_t < x_n$.

If $\limsup_{t\to\infty}k_t < x_n$, there exist a sufficiently large T and y_n such that $k_{T+t} < y_n < x_n$ for any $t \geq 0$. Hence, $f'(k_{T+t}) > f'(y_n) > f'(x_n) = 1$. By consequence, we obtain that $\lim_{T\to\infty}\frac{n^T}{\prod_{t=1}^T f'(k_t)}b_T = 0$.

Assume now that $\limsup_{t\to\infty}k_t \geq x_n$. Suppose that the solution is bubbly. According to point (iii) of Proposition 1, we have $\lim_{t\to\infty}\xi_t/b_t = 0$.

Let $\bar{x}$ satisfy $\liminf_{t\to\infty}k_t < \bar{x} < x_n$. Since $\limsup_{t\to\infty}k_t \geq x_n$, there exists T high enough satisfying $k_{T+1} \leq k_T$, $k_{T+1} \leq \bar{x}$ and $\xi_{T+t}/b_{T+t} \leq f'(\bar{x})/n - 1$ for any $t \geq 0$. For this T, we have

$$b_{T+1} = \frac{f'(k_{T+1})}{n}b_T - \xi_{T+1} \geq \frac{f'(\bar{x})}{n}b_T - \xi_T \geq b_T$$

and, therefore, $k_{T+2} = \mathcal{G}_x(k_{T+1}, b_{T+1}, \xi_{T+1}) \leq \mathcal{G}_x(k_T, b_T, \xi_T) = k_{T+1} < \bar{x}$. By induction, the sequence $(k_{T+t})_{t=0}^\infty$ is decreasing and converges to some value which is smaller that $\bar{x} < x_n$. This leads to a contradiction with the hypothesis $\limsup_{t\to\infty}k_t \geq x_n$. Hence, the solution is bubbleless.

We now prove that this is the unique equilibrium. Assume that there is another equilibrium $(k'_{t+1}, b'_t)_{t\geq 0}$ with $b'_0 > b_0$. Since $b'_0 > b_0$, we have $k'_t < k_t$ for any t. Hence, $\liminf_{t\to\infty}k'_t \leq \liminf_{t\to\infty}k_t = 0$. So, we get $\liminf_{t\to\infty}k'_t = 0$ which implies that the equilibrium $(k'_{t+1}, b'_t)_{t\geq 0}$ is bubbleless. This is impossible because according to point 3 of Lemma 3 there is at most one bubbleless equilibrium.

Point (iii) of Lemma 3 implies that the equilibrium is unique.

CASE 2: $\liminf_{t\to\infty}k_t \geq x_n$.

CASE 2.1. Focus first on the case $\liminf_{t\to\infty}k_t > x_n$. There exist $\epsilon > 0$ small and T high enough such that, for any $t \geq T$, we have $k_t > x_n + \epsilon$. This implies

$$b_{t+1} < \frac{f'(k_{t+1})}{n}b_t < \frac{f'(x_n+\epsilon)}{n}b_t.$$

Thus, the sequence (b_t) is decreasing and converges to 0.

Fix $b > 0$ and $\xi > 0$. Take T sufficiently high such that $b_{T+t} < b$, $\xi_{T+t} < \xi$ for any $t \geq 0$. Then, $\mathcal{G}^t_x(k_T, b, 0) \leq k_{T+t} \leq \mathcal{G}^t_x(k_T, 0, \xi)$ and, for any $b > 0$ and $\xi > 0$, $\liminf_{t\to\infty}k_t \geq k_{b,0}$ and $\limsup_{t\to\infty}k_t \leq k_{0,\xi}$.

Let b, ξ tend to 0 we get $\liminf_{t\to\infty}k_t = \limsup_{t\to\infty}k_t = k^*_x$.

CASE 2.2. Consider now the case $\liminf_{t\to\infty}k_t = x_n$. First, we prove that $\liminf_{t\to\infty}b_t \geq b_n$ where b_n satisfies $x_n = \mathcal{G}_x(x_n, b_n, 0)$. Suppose the contrary. Fix b such that $\liminf_{t\to\infty}b_t < b < b_n$. From $\mathcal{G}_x(k_{b,0}, b, 0) = k_{b,0}$ and $x_n < \mathcal{G}_x(x_n, b, 0)$, we get $k_{b,0} > x_n$. Since $\mathcal{G}_x(x_n, b, 0) > x_n$, we can take $\epsilon > 0$ satisfying $\mathcal{G}_x(x_n - \epsilon, b, 0) > x_n + \epsilon$. Take also T high enough such that $k_T > x_n - \epsilon$ and $b_T < b$. We find

$$k_{T+1} = \mathcal{G}_x(k_T, b_T, \xi_T) \geq \mathcal{G}_x(k_T, b, 0) > x_n + \epsilon$$
$$b_{T+1} = \frac{f'(k_{T+1})}{n}b_T - \xi_{T+1} < b_T < b.$$

By induction, we obtain $k_{T+t} > x_n + \epsilon$ and $b_{T+t} < b$ for any t. Hence, $\liminf_{t\to\infty}k_t \geq x_n + \epsilon > x_n$, that is a contradiction.

Since $\liminf_{t\to\infty}b_t \geq b_n$ for any $b < b_n$ and $\xi > 0$, there exists T satisfying $b_{T+t} > b$ and $\xi_{T+t} < \xi$ for any t. This implies $k_{T+t} < \mathcal{G}^t_x(k_T, b, \xi)$ and $\limsup_{t\to\infty}k_t \leq \mathcal{G}^t_x(k_T, b, \xi) = k_{b,\xi}$. Let b converge to b_n. ξ converges to 0. Thus, $k_{b,\xi}$ converges to $k_{b_n,0} = x_n$ and $\limsup_{t\to\infty}k_t \leq x_n \leq \liminf_{t\to\infty}k_t$. Hence, $\lim_{t\to\infty}k_t = x_n$ and $\lim_{t\to\infty}b_t = b_n$.

Remark 10. In the proof of cases $(2.b)$ and $(2.c)$ of Proposition 2, we do not use the monotonicity of $(\xi_t)_{t=0}^\infty$.

Appendix C. Proofs of Section 6

C.1. Proof of Proposition 3

Part 1. If $f'(k^*_x) > n$, Proposition 2 implies that there is a unique equilibrium and this equilibrium is bubbleless. Moreover, Proposition 2 also implies that b_t converges to zero. By consequence we have $\lim_{t\to\infty}b_t/(Ak^\alpha_t) = 0$ if $\liminf_{t\to\infty}k_t > 0$.

Part 2. We have only to consider the case $\liminf_{t\to\infty}k_t > 0$, or equivalently $\inf_t k_t > 0$.

Consider an equilibrium (b_t, k_{t+1}). Conditions (20) and (21) give

$$\frac{b_{t+1}}{Ak^\alpha_{t+1}} = \frac{\alpha b_t}{nk_{t+1}} - \frac{\xi_{t+1}}{Ak^\alpha_{t+1}}$$
$$= \frac{b_t}{Ak^\alpha_t}\frac{\alpha}{\alpha\gamma_x + (1-\sigma)\,\xi_t/\left(Ak^\alpha_t\right) - \sigma\, b_t/\left(Ak^\alpha_t\right)} - \frac{\xi_{t+1}}{Ak^\alpha_{t+1}}.$$

(1) Focus on the first case: there exists T such that $b_T/(Ak^\alpha_T) \leq \theta^*_x$. Then,

$$\frac{b_{t+1}}{Ak^\alpha_{t+1}} < \frac{b_t}{Ak^\alpha_t}\frac{\alpha}{\alpha\gamma_x + (1-\sigma)\,\xi_t/(Ak^\alpha_t) - \sigma\, b_t/(Ak^\alpha_t)}$$
$$< \frac{b_t}{Ak^\alpha_t}\frac{\alpha}{\alpha\gamma_x - \sigma\theta^*_x} = \frac{b_t}{Ak^\alpha_t} < \theta^*_x$$

for any $t \geq T$. The sequence $\left(b_t/(Ak^\alpha_t)\right)$ is decreasing. This implies the existence of $\lim_{t\to\infty}b_t/\left(Ak^\alpha_t\right) \equiv \theta$, with $0 \leq \theta < \theta^*_x$. Let us show that $\theta = 0$. Suppose that $\theta > 0$. θ becomes solution of $\theta = \alpha\theta/(\alpha\gamma_x - \sigma\theta)$ that is $\theta = \theta^*_x$: a contradiction. Thus, we have $\lim_{t\to\infty}b_t/(Ak^\alpha_t) = 0$. Since $\inf_t k_t > 0$, we get $\lim_{t\to\infty}b_t = 0$ and $\lim_{t\to\infty}k_t = k^*_x$.

(2) Focus on the second case: We have $b_t/(Ak^\alpha_t) > \theta^*_x$ for every t. Let us prove that $\lim_{t\to\infty}b_t/(Ak^\alpha_t) = \theta^*_x$. If the contrary holds, $\limsup_{t\to\infty}b_t/(Ak^\alpha_t) = \theta > \theta^*_x$ which implies in turn the existence

of $\varepsilon > 0$ and T high enough such that $b_T/(Ak_T^\alpha) > (1+\varepsilon)\,\theta_x^*$. Since $\xi_t \to 0$, we observe that

$$\lim_{t\to\infty} (1+\varepsilon)\,\theta_x^* \frac{\alpha}{\alpha\gamma_x + (1-\sigma)\,\xi_t/(Ak_t^\alpha) - \sigma\,(1+\varepsilon)\,\theta_x^*} - (1+\varepsilon)\,\theta_x^*$$
$$= (1+\varepsilon)\,\theta_x^* \frac{\alpha}{\alpha\gamma_x - \sigma\,(1+\varepsilon)\,\theta_x^*} - (1+\varepsilon)\,\theta_x^* > 0.$$

Thus, there exists T high enough such that $b_T/(Ak_T^\alpha) > (1+\varepsilon)\,\theta_x^*$ and, for every $t \ge T$,

$$\frac{\alpha\,(1+\varepsilon)\,\theta_x^*}{\alpha\gamma_x + (1-\sigma)\,\xi_t/(Ak_t^\alpha) - \sigma\,(1+\varepsilon)\,\theta_x^*} - (1+\varepsilon)\,\theta_x^* - \frac{\xi_{t+1}}{A(\inf_s k_s)^\alpha} > 0.$$

Therefore,

$$\frac{b_{T+1}}{Ak_{T+1}^\alpha} = \frac{b_T}{Ak_T^\alpha} \frac{\alpha}{\alpha\gamma_x + (1-\sigma)\,\xi_T/(Ak_T^\alpha) - \sigma b_T/(Ak_T^\alpha)} - \frac{\xi_{T+1}}{Ak_{T+1}^\alpha}$$
$$> \frac{\alpha\,(1+\varepsilon)\,\theta_x^*}{\alpha\gamma_x + (1-\sigma)\,\xi_T/(Ak_T^\alpha) - \sigma\,(1+\varepsilon)\,\theta_x^*} - \frac{\xi_{T+1}}{A(\inf_t k_t)^\alpha}$$
$$> (1+\varepsilon)\,\theta_x^*.$$

By induction, we find, for every $t \ge T$, $b_t/(Ak_t^\alpha) > (1+\varepsilon)\,\theta_x^*$ and

$$\frac{b_{t+1}}{Ak_{t+1}^\alpha} - \frac{b_t}{Ak_t^\alpha} = \frac{b_t}{Ak_t^\alpha} \frac{\alpha - \alpha\gamma_x - (1-\sigma)\,\xi_t/(Ak_t^\alpha) + \sigma b_t/(Ak_t^\alpha)}{\alpha\gamma_x + (1-\sigma)\,\xi_t/(Ak_t^\alpha) - \sigma b_t/(Ak_t^\alpha)} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha}$$
$$> (1+\varepsilon)\,\theta_x^* \times \frac{\alpha - \alpha\gamma_x - (1-\sigma)\,\xi_t/(Ak_t^\alpha) + \sigma\,(1+\varepsilon)\,\theta_x^*}{\alpha\gamma_x + \xi_t/(Ak_t^\alpha) - \sigma\,(1+\varepsilon)\,\theta_x^*} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha}$$
$$= (1+\varepsilon)\,\theta_x^* \frac{\sigma\varepsilon\theta_x^* - (1-\sigma)\,\xi_t/(Ak_t^\alpha)}{\alpha\gamma_x + \xi_t/(Ak_t^\alpha) - \sigma\,(1+\varepsilon)\,\theta_x^*} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha}.$$

This implies that $\liminf_{t\to\infty}\left[b_{t+1}/(Ak_{t+1}^\alpha) - b_t/(Ak_t^\alpha)\right] > 0$: for T high enough, the sequence $\left(b_t/(Ak_t^\alpha)\right)_{t=T}^{\infty}$ is increasing and converges to $\theta > \theta_x^*$. Applying the same argument of point (1), we get $\theta = \theta_x^*$, that is a contradiction.

It is immediate to see that $\lim_{t\to\infty} b_t/(Ak_t^\alpha) = \theta_x^*$ and, then, $k_t \to x_n$, and $b_t \to n\,(\gamma_x - 1)\,x_n/\sigma$ when t tends to infinity.

C.2. Proof of Example 1

The equilibrium system is written as

$$k_{t+1} = \frac{\alpha A\gamma_0 k_t^\alpha - b_t}{n}, \quad b_{t+1} = \frac{\alpha A b_t}{n k_{t+1}^{1-\alpha}} - \xi_{t+1},$$
$$b_t > 0,\ k_{t+1} > 0. \tag{C.1}$$

The proof is articulated in two steps.

STEP 1. Let (x_t) be a positive sequence such that

$$x_t \ge \frac{1}{\gamma_0} \text{ and } x_t + \frac{1}{\gamma_0 x_{t+1}} \ge \frac{1}{\gamma_0} + 1 \tag{C.2}$$

for every t (such a sequence exists, for example, $x_t = 1/\gamma_0$ for any t). We prove that there exist a sequence of nonnegative dividends (ξ_t) and $\left(\bar{b}_t, \bar{k}_{t+1}\right)$ a solution of system (C.1) with

$$\bar{k}_{t+1} = \frac{\alpha A \bar{k}_t^\alpha}{n x_t} \forall t. \tag{C.3}$$

To show that such sequences exist, consider the sequence $\left(\bar{b}_t, \bar{k}_{t+1}\right)$ defined by (C.3) and $\bar{b}_t = \alpha A\gamma_0 \bar{k}_t^\alpha - n\bar{k}_{t+1}$. Since $x_t \ge 1/\gamma_0$, we have $\bar{b}_t = \alpha A\gamma_0\bar{k}_t^\alpha - n\bar{k}_{t+1} \ge 0$ for every t. We define the sequence (ξ_t) with dividends (23) for every $t \ge 0$. Then,

$$\xi_{t+1} = \frac{\alpha A\left(\alpha A\gamma_0\bar{k}_t^\alpha - n\bar{k}_{t+1}\right)}{n\bar{k}_{t+1}^{1-\alpha}} - \left(\alpha A\gamma_0\bar{k}_{t+1}^\alpha - n\bar{k}_{t+2}\right)$$
$$= \alpha A\gamma_0\bar{k}_{t+1}^\alpha\left(\frac{\alpha A\bar{k}_t^\alpha}{n\bar{k}_{t+1}} - \frac{1}{\gamma_0} - 1 + \frac{n\bar{k}_{t+2}}{\alpha A\gamma_0\bar{k}_{t+1}^\alpha}\right)$$
$$= \alpha A\gamma_0\bar{k}_{t+1}^\alpha\left(x_t + \frac{1}{\gamma_0 x_{t+1}} - 1 - \frac{1}{\gamma_0}\right). \tag{C.4}$$

According to inequality (C.2), we see that $\xi_{t+1} \ge 0$ for every $t \ge 0$ and, therefore, $\left(\bar{b}_t, \bar{k}_{t+1}\right)$ is solution of system (C.1) with sequence of dividends (ξ_t).

STEP 2. Let us now prove Example 1. We see that $x_t = e^{\lambda^t}$ for every $t \ge 1$, and $x_0 \equiv \max\{e, 1 + 2/\gamma_0\} \ge e$. The sequence (x_t) satisfies restrictions (C.2). Consider the sequence $(\bar{b}_t, \bar{k}_{t+1})$ defined by (C.3) and $\bar{b}_t = \alpha A\gamma_0\bar{k}_t^\alpha - n\bar{k}_{t+1}$ jointly with the sequence of dividends (23). We have $\lim_{t\to\infty} x_t = \infty$ and, according to (C.4), $\xi_{t+1} > 0$ for every t. Thus, the sequence $\left(\bar{b}_t, \bar{k}_{t+1}\right)$ is solution of system (C.1) with $\lim_{t\to\infty}\bar{k}_t = \lim_{t\to\infty}\bar{b}_t = 0$.

Let us prove now that $\lim_{t\to\infty}\xi_{t+1} = 0$. According to (C.4) and the fact that $\lim_{t\to\infty} x_t = \infty$, it is sufficient to prove that $\lim_{t\to\infty}\bar{k}_{t+1}^\alpha x_t = 0$. Solving recursively (C.3), and using $x_t = e^{\lambda^t}$, we find that

$$\bar{k}_{t+1}^\alpha = \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{\bar{k}_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} \Pi_{s=0}^{t-1} \frac{1}{x_{t-s}^{\alpha^{1+s}}}$$
$$= \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{\bar{k}_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} e^{-\sum_{s=0}^{t-1}\alpha^{1+s}\lambda^{t-s}}.$$

We notice that λ is solution of $\lambda^t = \sum_{s=0}^{2}\alpha^{1+s}\lambda^{t-s}$. Then, for $t > 4$,

$$\bar{k}_{t+1}^\alpha x_t < \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{\bar{k}_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} e^{-\sum_{s=0}^{3}\alpha^{1+s}\lambda^{t-s}} x_t$$
$$= \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{\bar{k}_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} e^{\lambda^t - \sum_{s=0}^{3}\alpha^{1+s}\lambda^{t-s}}$$
$$= \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{\bar{k}_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} e^{-\alpha^4\lambda^{t-3}}.$$

Since $\lambda > 1$, we get that $\lim_{t\to\infty}\bar{k}_{t+1}^\alpha x_t \le 0$.

By using (C.4) and the fact that $\lim_{t\to\infty}\bar{k}_{t+1} = 0$, $\lim_{t\to\infty} x_t = \infty$, it is easy to check that $\lim_{t\to\infty}\xi_t/\bar{k}_t = \infty$ and $\lim_{t\to\infty}\xi_t/f(\bar{k}_t) = \infty$.

C.3. Proof of Example 2

Let (b_t, k_{t+1}) be a sequence determined by the system (20)–(21) and $b_0 \in \left(\theta_x^* Ak_0^\alpha/n, \theta_x^* Ak_0^\alpha\right)$. To prove that this is an equilibrium, we check that $b_t > 0$ and $k_{t+1} > 0$ for any $t \ge 0$.

According to (20), we see that $k_{t+1} > 0$ if $\alpha A\gamma_x k_t^\alpha > \sigma b_t$, which is satisfied when $b_t < \theta_x^* Ak_t^\alpha$. Therefore, we have just to prove that $b_t \in \left(\theta_x^* Ak_t^\alpha/n^{t+1}, \theta_x^* Ak_t^\alpha\right)$ for every t. Let us apply the induction argument.

(1) We show first that $b_t < \theta_x^* A k_t^\alpha$ implies $b_{t+1} < \theta_x^* A k_{t+1}^\alpha$. Indeed, considering (20) and (21), we find

$$\begin{aligned}\frac{b_{t+1}}{Ak_{t+1}^\alpha} &= \frac{b_t\alpha}{Ak_t^\alpha \alpha\gamma_x + (1-\sigma)\xi_t - \sigma b_t} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha}\\ &= \frac{b_t}{Ak_t^\alpha}\frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi_t - \sigma b_t}{Ak_t^\alpha}} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha}\\ &\le \frac{b_t}{Ak_t^\alpha}\frac{\alpha}{\alpha\gamma_x - \frac{\sigma b_t}{Ak_t^\alpha}} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha}\\ &< \frac{b_t}{Ak_t^\alpha}\frac{\alpha}{\alpha\gamma_x - \sigma\theta_x^*} = \frac{b_t}{Ak_t^\alpha} \le \theta_x^*.\end{aligned}$$

(2) Then, we prove that $b_t > \theta_x^* A k_t^\alpha / n^{t+1}$ implies $b_{t+1} > \theta_x^* A k_{t+1}^\alpha / n^{t+2}$. Since $b_t \le \theta_x^* A k_t^\alpha$ for every t, we have $k_t \ge k_m$ for every t. Using (20) and (21), we obtain

$$\begin{aligned}\frac{b_{t+1}}{Ak_{t+1}^\alpha} - \frac{\theta_x^*}{n^{t+2}} &= \frac{b_t}{Ak_t^\alpha}\frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi_t - \sigma b_t}{Ak_t^\alpha}} - \frac{\xi}{n^{t+1}Ak_{t+1}^\alpha} - \frac{\theta_x^*}{n^{t+2}}\\ &> \frac{\theta_x^*}{n^{t+1}}\frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi}{n^t Ak_t^\alpha} - \frac{\sigma\theta_x^*}{n^{t+1}}} - \frac{\xi}{n^{t+1}Ak_{t+1}^\alpha} - \frac{\theta_x^*}{n^{t+2}}\\ &> \frac{\theta_x^*}{n^{t+1}}\left[\frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi}{n^t Ak_t^\alpha}} - \frac{1}{n} - \frac{\xi}{\theta_x^* Ak_{t+1}^\alpha}\right]\\ &\ge \frac{\theta_x^*}{n^{t+1}}\left[\frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi}{Ak_m^\alpha}} - \frac{1}{n} - \frac{\xi}{\theta_x^* Ak_m^\alpha}\right] > 0.\end{aligned}$$

C.4. Proof of Proposition 4

A bubble exists if and only if $b_t > 0$ for any t.

Combining (25) and (26), we get a single dynamic equation:

$$z_{t+1} = \gamma_x z_t - 1 \quad \forall t \ge 0 \tag{C.5}$$

where $z_t \equiv nk_{t+1}/(\sigma b_t)$. The solution of the difference Eq. (C.5) is given by $z_t = \gamma_x^t z_0 - \frac{1-\gamma_x^t}{1-\gamma_x}$, $\forall t \ge 1$, provided that $\gamma_x \ne 1$.

(1) When $\gamma_x \le 1$, there is no bubble. Indeed, if $\gamma_x \le 1$, z_t becomes negative soon or later: this leads to a contradiction. In this case, capital transition becomes $k_{t+1} = \rho_{\gamma_x} k_t^\alpha$. Solving recursively, we find the explicit solution (29). We observe that, according to (27), $\lim_{t\to\infty} k_t = \rho_{\gamma_x}^{1/(1-\alpha)} = k_x^*$.

(2) Let $\gamma_x > 1$.

(2.a) If $b_t = 0$, then (29) follows immediately.

(2.b) Focus on the case $b_t > 0$. Then, we obtain

$$z_t = \frac{[(\gamma_x - 1) z_0 - 1]\gamma_x^t + 1}{\gamma_x - 1}. \tag{C.6}$$

A positive solution exists if and only if $z_0 \ge 1/(\gamma_x - 1)$. Hence, the existence of a positive solution requires

$$b_0 \le \frac{\gamma_x - 1}{\sigma} nk_1 = \frac{\gamma_x - 1}{\sigma}\left[\frac{\beta}{1+\beta}(w_0 + g_0) - b_0\right].$$

Solving this inequality for b_0, we find $0 < b_0 \le \bar{b}_x$.

Now, given $b_0 \in (0, \bar{b}_x]$, the sequence (k_{t+1}, b_t) constructed by (25) and (26) is an equilibrium with $b_t > 0$ for any t.

When $b_0 < \bar{b}_x$ (that is $z_0 > 1/(\gamma_x - 1)$), because of (C.6), we get $\lim_{t\to\infty} z_t = \infty$. According to (25), k_t is uniformly bounded from above, which implies that $\lim_{t\to\infty} b_t = 0$. Thus, $\lim_{t\to\infty} k_t = k_x^*$.

When $b_0 = \bar{b}_x$, we have $z_t = 1/(\gamma_x - 1)$ for any $t \ge 0$. In this case, $k_{t+1} = \rho_1 k_t^\alpha$ where $\rho_1 \equiv \alpha A/n$ for any $t > 0$ and $b_t = (\gamma_x - 1) nk_{t+1}/\sigma$. Solving recursively, we get the explicit solution (30).

C.5. Proof of Proposition 5

(1) When $R > n$.

Denote $D \equiv \frac{R}{n}\frac{\beta}{1+\beta}\frac{x}{1+x}$. According to (34), and using the fact that $b_t + \xi_t = \frac{R}{n} b_{t-1}$ we have

$$\begin{aligned}k_{t+1} &= Dk_t + \frac{1}{n}\frac{\beta}{1+\beta}\Big(w + \frac{x}{1+x}\xi_t\Big)\\ &\quad - \frac{1}{n}\Big(1 - \frac{\beta}{1+\beta}\frac{x}{1+x}\Big)b_t &\text{(C.7)}\\ &= Dk_t + \frac{1}{n}\frac{\beta}{1+\beta}w + \frac{1}{n}(Db_{t-1} - b_t). &\text{(C.8)}\end{aligned}$$

So, we find that $nk_{t+1} + b_t = D(nk_t + b_{t-1}) + \frac{\beta w}{1+\beta}$, and therefore

$$\begin{aligned}\frac{nk_{t+1} + b_t}{D^t} &= \frac{nk_t + b_{t-1}}{D^{t-1}} + \frac{\beta w}{1+\beta}\frac{1}{D^t} \implies \frac{nk_{t+1} + b_t}{D^t}\\ &= nk_1 + b_0 + \frac{\beta w}{1+\beta}\sum_{s=1}^{t}\frac{1}{D^s}\\ \implies nk_{t+1} + b_t &= D^t(nk_1 + b_0) + \frac{\beta w}{1+\beta}\sum_{s=0}^{t-1} D^s\\ &= D^t(nk_1 + b_0) + \frac{\beta w}{1+\beta}\frac{D^t - 1}{D - 1}. &\text{(C.9)}\end{aligned}$$

Since $nk_1 + b_0 = \frac{\beta}{1+\beta}(w + g_0)$, we obtain

$$nk_{t+1} = D^t(nk_1 + b_0) + \frac{\beta w}{1+\beta}\frac{D^t - 1}{D - 1} - \frac{R^t}{n^t}\Big(b_0 - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\Big).$$

If there is an equilibrium with bubble, then $b_0 > \sum_{s=1}^{\infty}\frac{n^s}{R^s}\xi_s$. Let us denote $B := b_0 - \sum_{s=1}^{\infty}\frac{n^s}{R^s}\xi_s$. Since $D < R/n$, we observe that $b_t > \left(\frac{R}{n}\right)^t B$, which converges to infinity and grows faster than the right hand side of (C.9). Hence, k_{t+1} will be strictly negative for t high enough, a contradiction. Hence, there is no bubble.

(2) When $R \le n$. The proof in this case is easy.

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