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An investigation of model risk in a market with jumps and stochastic volatility

Guillaume Coqueret[a], Bertrand Tavin[b,*]

[a] Montpellier Business School, 2300 Avenue des Moulins, 34080 Montpellier, France
[b] EMLYON Business School, 23 Avenue Guy de Collongue, 69130 Ecully, France

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ABSTRACT

The aim of this paper is to investigate model risk aspects of variance swaps and forward-start options in a realistic market setup where the underlying asset price process exhibits stochastic volatility and jumps. We devise a general framework in order to provide evidence of the model uncertainty attached to variance swaps and forward-start options. In our study, both variance swaps and forward-start options can be valued by means of analytic methods. We measure model risk using a set of 21 models embedding various dynamics with both continuous and discontinuous sample paths. To conduct our empirical analysis, we work with two major equity indices (S&P 500 and Eurostoxx 50) under different market situations. Our results evaluate model risk between 50 and 200 basis points, with an average value slightly above 100 basis points of the contract notional.

1. Introduction

Because of their highly customizable features, derivative contracts require that their seller determines a price which is at the same time profitable for the issuer and attractive for the client. Very often, these products are evaluated as the sum of a fair price plus a margin. While this margin is left at the appreciation of the agent selling the product, the fair price is usually determined as the expected discounted payoff of the derivative under a suitable probability. In order to do so, a widespread approach is to specify a parametric model for the underlying (or underlyings) pertaining to the derivative. Unfortunately, two different models, or one model with two parametrizations, will generate different prices. The uncertainty relative to the choice of a model implies a risk for the issuer of the derivative which is referred to as model risk. The measurement of this risk is a crucial question faced by financial institutions.

In the present paper, we focus on variance swaps and forward-start options. The reason underpinning this choice is twofold. First, these instruments are particularly popular among clients of financial institutions. Hence, they represent an important part of their derivatives trading books and eventually an exposure to model risk. Second, as we will show subsequently, variance swaps and forward-start options can be priced using analytical methods in a wide range of models. This makes them suitable to be studied with the chosen methodology that relies on repetitive pricing and calibration.

In practice, the pricing of complex financial products (such as those studied in this paper) requires three steps (see also Fig. 1):

1. Choice of a stochastic model for the dynamics of the underlying variables (e.g. a parametric model built with stochastic processes).
2. Calibration of the chosen model using market data (the model is expected to yield prices which are as close as possible to those observed on the market, otherwise it is better not to use it).
3. Pricing of the complex product with numerical methods (Monte-Carlo methods, numerical resolution of partial differential equations, numerical integration, etc.).

The main purpose of the present paper is to show that the first step can have a considerable impact on the final outcome of the pricing process. It means that a bank relying on such a process is exposed to a risk during Step 1 and we aim at measuring this risk.

In a realistic market setup, the value of a variance swap is model-dependent even if vanilla option (i.e. classical, liquid and publicly traded) quotes are available for all strikes and maturities. Hence, the question of the model uncertainty affecting the value of a variance swap becomes of great interest. A lot of model uncertainty means that two different models can lead to two very different prices. Now, since models have been calibrated with

* Corresponding author. Tel.: +33 478337800; fax: +33 478336169.
E-mail addresses: g.coqueret@montpellier-bs.com (G. Coqueret), tavin@em-lyon.com (B. Tavin).

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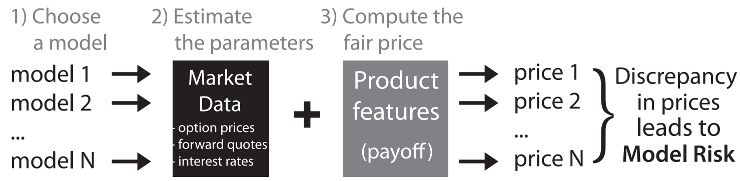

Fig. 1. Steps for pricing path-dependent financial products.

option data, substantial model risk for variance swaps means that the market data was not able to sufficiently characterize the distribution of the dynamics of the underlying. This is very valuable information because if some variance swap prices are publicly available, then they can be used to complete the characterization of the dynamics of the underlying. This implies that these prices can be included in the second step (calibration process) when pricing other complex derivatives.

Forward-start options are the other type of financial instruments which we will consider in our study. Forward-start options are known for being difficult to handle due to their high sensitivity to the choice of a model. Hence, the study of the model risk affecting these instruments and its possible reduction (with variance swap quotes) are of the utmost interest.

Our paper strongly relates to the literature on model risk, which dates back to Derman (1996) in the field of finance. Our approach is inspired by Cont (2006), but in contrast to this paper, we aim at an empirical quantification of model risk, as opposed to an essentially theoretical one. In the field of operational research, our approach can be linked to the broader family of approaches based on *worst-case scenario analysis*; see for example Kapsos, Christofides, and Rustem (2014) and Gülpinara and Rustem (2007). Numerous articles have already dealt with model risk (we refer to Section 5.1 for more details), but none of them relies on a systematic evaluation of a large set of models, which is the core principle of our methodology.

The main contributions of this paper are the following:

- We provide the first study on model risk based on a large scale of models (i.e. more than the three to five used in previous articles). Moreover, the prices for variance swaps and forward-start options that we use are realistic because they stem from accurately calibrated models. We find that model risk lies around 100 *basis points* of value, irrespectively of the maturity of the variance swap.
- We test the robustness of our result in two directions: we show that model risk remains unchanged even when variance swaps are hedged with popular strategies; and we also observe that model risk can be reduced when prices of path-dependent products are added to the calibration set. This possibility to reduce model risk is of interest for banks handling these products.
- We highlight that model risk strongly depends on the variety of models taken into account. Both continuous and discontinuous models should be included to avoid an underestimation of model risk. For models built with Lévy processes, we find that the choice relative to each component creates a comparable amount of risk.
- Finally, we provide analytical results for the pricing of variance swaps and forward-start options that were not yet available for some of the considered models.

The remainder of the paper is organized as follows. In Section 2 we describe our financial framework and probabilistic setting. In Section 3 we review the set of models used in the paper. In Section 4 we review the problem of valuing variance swaps and forward-start options in our framework, while providing en route some new analytical pricing formulae. In Section 5 we detail the design of our empirical study and discuss the obtained results. Section 6 concludes. A companion supplementary materials appendix provides additional tables, proofs and technical comments.

2. The financial framework

In this section we devise the general financial framework in which variance swaps and forward-start options are considered. The characteristics of our market model are given. We also detail the stochastic setup in which the underlying asset dynamics will be modeled.

2.1. Setting and assumptions

We consider a fundamental filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ with $\mathbb{P}$ the historical probability measure and $T < +\infty$ a final time horizon. The price of the risky (underlying) asset is modeled by the stochastic process $(S_t)_{t\,\in\,[0,\,T]}$ that is adapted to $(\mathcal{F}_t)_{t=0}^T$ (trading takes place in continuous time). This risky asset pays a continuous dividend denoted by d and this dividend is assumed to be constant. The risk-free rate is also constant and equal to $r > 0$. It will be used to discount future cash-flows (payoffs in our setting). The current value of the risky asset is known and denoted by S_0. The market is considered arbitrage-free and incomplete, hence there exists at least one equivalent martingale measure[1] $\mathbb{Q}$, under which the discounted price process with reinvested dividends $(e^{-rt}e^{dt}S_tS_t)_{t\in[0,T]}$ is a martingale. We will henceforth work only under $\mathbb{Q}$. Consequently, we have

$$S_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}e^{dT}S_T\right]. \tag{1}$$

Let Z be a contingent claim with a single payoff paid at time T and denoted by Z_T. Z_T is a random variable on $(\Omega, \mathcal{F}_T, \mathbb{P})$. Z is said to be path-dependent if its payoff function, denoted by z and known at inception, is a function of the trajectory of S on $[0, T]$, that is if $Z_T = z(\{S_t, t \in [0, T]\})$. At initial time, a no arbitrage price of Z is given by

$$Z_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}z(\{S_t, t \in [0, T]\})\right]. \tag{2}$$

When the payoff is a function of the final asset price only, the contingent claim is said to be European and $Z_T = z(S_T)$. We assume that vanilla options (calls and puts) written on S are available for a finite number of strikes and maturities. $C(K, T)$ denotes the call option struck at K and with maturity T. Its payoff is written

$$C_T(K, T) = (S_T - K)^+ = \max(S_T - K, 0),$$

and its initial price is $C_0(K, T)$. With the same notations, $P(K, T)$ denotes the put option with strike K and maturity T. It is important to understand the implications of Eq. (2). If a public price Z_0 is

[1] $\mathbb{Q}$ is usually referred to as a risk-neutral probability because it corresponds to a probability that would prevail in an artificial world where economic agents are neutral towards risk.

available for the payoff z, then this can be seen as valuable information about the dynamics of the underlying S under $\mathbb{Q}$. Indeed, if many calls and puts have quotes for maturity T then these prices can be used to determine the shape of the distribution of S_T, via Eq. (2). Only a few distributions can correctly match market prices, see for instance Jackwerth and Rubinstein (1996). Now, even more interestingly, when working with path-dependent options, the information embedded in the prices can be used to characterize not only the distribution of S at a fixed time horizon, but also *through* time. This is because the payoff depends on the whole trajectory of the underlying until maturity.

We denote by Y the log-return of the risky asset, so that we have $S_t = S_0 e^{Y_t}$. It is noteworthy to remark that in order for the discounted price $e^{(d-r)t}S_t$ to be a $\mathbb{Q}$-martingale, the stochastic process Y should be such that, for $t \in [0, T]$,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{Y_t}\right] = e^{(r-d)t}. \tag{3}$$

We will refer to this constraint as the martingale constraint. The general formulation of the models we use in the sequel is done by means of characteristic functions (hereafter CF) that specifies the risk-neutral dynamics of $(Y_t)_{t\in[0,T]}$. Ψ will denote the CF of Y and is written, for $u \in \mathbb{C}$, $\Psi(u,t) = \mathbb{E}^{\mathbb{Q}}[e^{iuY_t}]$. For more details on characteristic functions and the martingale constraint we refer to Schoutens (2003) and Cont and Tankov (2004).

2.2. Log-return dynamics

In this paper, we consider parametric dynamics of Y that can be gathered in two groups. For models in the first group (Group 1), the dynamics of Y is directly modeled under $\mathbb{Q}$, using a stochastic differential equation (SDE). This SDE has a diffusion component and an eventual jump component. For models in this group, the volatility of the diffusion component is stochastic and is driven by another SDE.

For models in the second group (Group 2), we rely on geometric Lévy processes to describe the dynamics of Y and stochastic volatility is introduced by means of a random time change. For models in this group, the dynamics of the log-return Y under $\mathbb{Q}$ is written

$$Y_t = \ln\left(\frac{S_t}{S_0}\right) = (r-d)t + X_{\tau(t)} - \omega_\tau(t), \tag{4}$$

$$\omega_\tau(t) = \ln \mathbb{E}^{\mathbb{Q}}\left[e^{X_{\tau(t)}}\right], \tag{5}$$

where X is a driving Lévy process and the stochastic change of time is introduced via τ. $\tau(t)$ is the business time associated with calendar time t. The presence of $\omega_\tau(t)$ in the dynamics ensures that the constraint (3) is met.

A Lévy process X is a stochastic process with independent and identically distributed increments. The characteristic exponent of X_t at any time $t \in [0, T]$ is written

$$\phi_X(u,t) = \ln \mathbb{E}^{\mathbb{Q}}\left[e^{iuX_t}\right] = t\phi_X(u,1). \tag{6}$$

The dynamics of X is usually characterized by the Lévy–Khintchine representation of $\phi_X(u, 1)$, the characteristic exponent of X_1. For more details on Lévy processes and the Lévy–Khintchine representation, we refer to Schoutens (2003) and Cont and Tankov (2004).

A stochastic time change τ is defined as the integral of a positive stochastic process y known as the rate of time change. It is used to introduce time-varying trading activity in a stochastic model, the idea being to mimic what is observed in practice. If y is the constant 1, the business time is identical to the calendar time. Given y_0, $\tau(t)$ is defined as

$$\tau(t) = \int_0^t y_s ds. \tag{7}$$

Henceforth, and without loss of generality, we will consider $y_0 = 1$. This approach of stochastic clocks for Lévy processes has been introduced in (Carr, Geman, Madan, & Yor, 2003). The dynamics of y under $\mathbb{Q}$ is chosen to be a stationary mean-reverting process. The stationarity of y implies that the time-t_0 characteristic exponent of $\tau(t)-\tau(t_0)$, given y_{t_0}, satisfies

$$\begin{aligned}\phi_\tau(u, t-t_0, y_{t_0}) &= \ln \mathbb{E}^{\mathbb{Q}}\left[e^{iu(\tau(t)-\tau(t_0))}\middle| y_{t_0}\right] \\ &= \ln \mathbb{E}^{\mathbb{Q}}\left[\exp\left(iu\int_{t_0}^t y_s ds\right)\middle| y_{t_0}\right].\end{aligned} \tag{8}$$

We assume that the characteristic exponent of $\tau(t)$ seen at time 0 and given y_0 can be decomposed as

$$\phi_\tau(u,t,y_0) = ig(u,t)y_0 + h(u,t). \tag{9}$$

This decomposition corresponds to the time changes used in the sequel and will reveal itself particularly useful for the pricing of forward-start options.

A model in the second group is built by choosing a driving Lévy process X and a stochastic clock τ. Its dynamics is characterized by the CF of the log-return Y_t that as the generic form

$$\begin{aligned}\Psi(u,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{iuY_t}\right] = \exp[iu((r-d)t \\ - \omega_\tau(t)) + \phi_\tau(-i\phi_X(u,1),t,1)],\end{aligned} \tag{10}$$

$$\omega_\tau(t) = \ln \mathbb{E}^{\mathbb{Q}}\left[e^{X_{\tau(t)}}\right] = \phi_\tau(-i\phi_X(-i,1),t,1), \tag{11}$$

where ϕ_X and ϕ_τ are the characteristic exponents of X and τ, respectively.

2.3. Forward characteristic exponent

For the pricing of forward-start options, the knowledge of the forward characteristic exponent (hereafter FCE) will also be useful. The FCE is defined as the characteristic exponent of the forward log-return $\ln\frac{S_T}{S_{T_0}} = Y_T - Y_{T_0}$, with $0 < T_0 < T < +\infty$, seen from current time $t = 0$. It is defined as

$$\phi_{T_0,T}(u) = \ln \mathbb{E}^{\mathbb{Q}}\left[e^{iu\ln\frac{S_T}{S_{T_0}}}\right] = \ln \mathbb{E}^{\mathbb{Q}}\left[e^{iu\left(Y_T - Y_{T_0}\right)}\right]. \tag{12}$$

In our framework, the general approach to derive $\phi_{T_0,T}(u)$ is to resort to double conditioning using the independence of increments:

$$\phi_{T_0,T}(u) = \ln \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[e^{iu(Y_T-Y_{T_0})}|Y_{T_0}]]. \tag{13}$$

This approach is suitable for all the models considered in the sequel. In particular, it can be used to get the FCE of models built as geometric Lévy processes with changes of time (Group 2). Let X be the chosen driving Lévy process and τ the chosen change of time. The FCE can be decomposed as

$$\begin{aligned}\phi_{T_0,T}(u) &= \ln \mathbb{E}^{\mathbb{Q}}[e^{iu(Y_T-Y_{T_0})}] \\ &= iu[(r-d)\Delta T - \phi_\tau(-i\phi_X(-i,1),T,1) \\ &\quad + \phi_\tau(-i\phi_X(-i,1),T_0,1)] \\ &\quad + \phi_y(g(-i\phi_X(u,1),\Delta T),T_0,1) + h(-i\phi_X(u,1),\Delta T),\end{aligned} \tag{14}$$

with $\Delta T = T - T_0$ and $\phi_y(u, T_0, 1)$ the characteristic exponent of y_{T_0} taken at u and given $y_0 = 1$. The proof of this decomposition can be found in the supplementary materials appendix.

3. Description of the set of models

In this section, we review the set of 21 models we use in the paper to investigate model risk. As we have already mentioned, this set is separated in two groups. The list of models is gathered in Table 1.

For models in Group 1, the dynamics of Y is directly specified as a SDE with stochastic volatility and an eventual jump component. In the Heston model (H), the risk-neutral dynamics of the

Table 1
List of all models tested in the study. Models for which we provide new closed-form expressions for the pricing of variance swaps and forward-start options are identified with a ⋆ and a ⋄, respectively.

No.	Acronym	Group	No.	Acronym	Group	No.	Acronym	Group
1	H	1	8	CGMY-OUG⋄	2	15	Meix-OUIG⋆, ⋄	2
2	DH	1	9	CGMY-OUIG⋄	2	16	NIG-CIR	2
3	HJ	1	10	Kou-CIR	2	17	NIG-OUG⋄	2
4	VG-CIR	2	11	Kou-OUG⋄	2	18	NIG-OUIG⋄	2
5	VG-OUG⋄	2	12	Kou-OUIG⋄	2	19	GH-CIR⋆	2
6	VG-OUIG⋄	2	13	Meix-CIR⋆	2	20	GH-OUG⋆, ⋄	2
7	CGMY-CIR	2	14	Meix-OUG⋆, ⋄	2	21	GH-OUIG⋆, ⋄	2

Table 2
List of stochastic processes related to the second group of models. Panel A (*top*) presents the driving Lévy processes and Panel B (*bottom*) presents the stochastic clocks. For each process in the table we provide its usual name, the acronym used in this paper as well as the main references.

Acronym	Name	References
Panel A: Driving Lévy processes		
VG	Variance Gamma	Madan and Seneta (1990)
CGMY	CGMY	Carr, Geman, Madan, and Yor (2002)
Kou	Double exponential jump-diffusion	Kou (2002) and Kou and Wang (2004)
Meix	Meixner	Schoutens and Teugels (1998)
NIG	Normal Inverse Gaussian	Barndorff-Nielsen (1997), 1998)
GH	Generalized Hyperbolic	Barndorff-Nielsen (1977)
Panel B: Stochastic clocks		
CIR	Integrated CIR	Carr et al. (2003) and Cox et al. (1985)
OUG	Integrated OU-Gamma	Barndorff-Nielsen and Shephard (2001)
OUIG	Integrated OU-Inverse Gaussian	Barndorff-Nielsen and Shephard (2001)

underlying asset price is defined by means of a system of SDE built on two correlated Brownian motions; see Heston (1993). The Double Heston model (DH) has been introduced as a two-factor generalization of (H) by Christoffersen, Heston, and Jacobs (2009). Another way to generalize Heston's model is to introduce independent jumps in the dynamics of the underlying asset price. The Heston's model with jumps (HJ) we consider is the version introduced in (Bates, 1996) and (Bakshi, Cao, & Chen, 1997). In this version the jump component is built with a Poisson process. The expression of the CF we retain for (H) is not the expression proposed in the original paper but is more convenient for numerical applications; see del Baño Rollin, Ferreiro-Castilla, and Utzet (2010). Under (DH) the CF of the log-return Y_t is a straightforward two-factor generalization of the CF under (H), we refer to Christoffersen et al. (2009) and Gauthier and Possamai (2011) for a proof. Under (HJ), Bates (1996) and Bakshi et al. (1997) obtain the CF of the log-return as an extension of the CF under (H). The expressions of the CF for models in Group 1 are gathered in the supplementary materials appendix. The expressions of the FCE for these models can be found in Hong (2004) and Kruse and Nogel (2005), they are collected in the supplementary materials appendix.

Group 2 encompasses 18 models built on the basis of geometric Lévy processes with stochastic volatility introduced as a random time change. They are listed in Table 1. We consider six possible driving Lévy processes to specify the dynamics of Y. Table 2 provides the list of these processes, as well as the principal references and the corresponding acronyms. We consider three stochastic time changes. The dynamics of the time change rate y, under $\mathbb{Q}$, is chosen to be a stationary mean-reverting process (of CIR or Ornstein–Uhlenbeck type). Table 2 provides the list of these clocks, as well as the principal references and the corresponding acronyms.

The expressions of the characteristic exponents ϕ_X for the chosen Lévy processes are gathered in the supplementary materials appendix. For the chosen stochastic clocks, the decompositions of ϕ_τ according to Eq. (9) are presented in the supplementary materials appendix. These characteristic exponent are obtained from the dynamics of the time change rate y that is usually given by its CF Ψ_y. For the CIR stochastic clock, the computation of the characteristic exponent can be found in (Cox, Ingersoll, & Ross, 1985). For the OUG and OUIG stochastic clocks, characteristic exponents are computed in (Schoutens, 2003).

Table 3
Characteristic exponents for time change rates, seen at time 0 and given y_0. Stochastic clocks identified with a ⋄ are those for which the presented formula is obtained in this paper. We refer to the supplementary materials appendix for proofs.

Change of time	Characteristic exponent of y
ICIR	$\phi_y(u,t,y_0) = -\frac{2\kappa\eta}{\lambda^2}\ln\left[1-\frac{\lambda^2 iu}{2\kappa}\left(1-e^{-\kappa t}\right)\right]+\frac{iue^{-\kappa t}y_0}{1-\frac{\lambda^2 iu}{2\kappa}(1-e^{-\kappa t})}$
OUG⋄	$\phi_y(u,t,y_0) = iue^{-\lambda t}y_0 + a\ln\frac{b-iue^{-\lambda t}}{b-iu}$
OUIG⋄	$\phi_y(u,t,y_0) = iue^{-\lambda t}y_0 + a\left(\sqrt{b^2-2iue^{-\lambda t}}-\sqrt{b^2-2iu}\right)$

For models in Group 2, the generic decomposition of the FCE is given by expression (14) that involves the characteristic exponent ϕ_y for the chosen time change rate. Table 3 gathers the expressions of the characteristic exponents ϕ_y for the chosen time change rates. For the ICIR time change, the characteristic exponent of y_t, seen from time 0 and given y_0 can be deduced from the characteristic function of the short rate in the CIR model for interest rates. For more details, we refer to the original article Cox et al. (1985). For OUG and OUIG, to the best of our knowledge, expressions are not available in the previous literature and we give the proofs in the supplementary materials appendix.

4. Variance swaps and forward-start options

In this section, we describe the exotic derivative contracts under scrutiny in this paper, namely variance swap contracts and forward-start options. We briefly explain their definition and functioning. We recall that these two types of products were chosen because they are quite popular among investment banks and their customers. As such they can represent an exposure to model risk. In this section we explain that these products can be valued by highly efficient numerical methods. Hence Step 3 defined in the introduction is not an issue when repeated for several models. Additional details and results are provided in the supplementary materials appendix.

4.1. Variance swap contracts

A variance swap is a financial contract written on the realized variance of the underlying asset price during a reference period. It is usually traded over-the-counter (that is, privately between two parties). The common practice is to monitor the asset price on a daily basis in order to compute the realized variance. In this paper we consider variance swaps to be continuously monitored by assuming that the difference with actual contracts is small. For a discussion relative to the link between discretely and continuously monitored variance swaps, we refer to Bernard and Cui (2014) and references therein.

Let $[0, T]$ be the contract reference period. We denote by $[Y]_T = \langle Y, Y\rangle_T$ the realized quadratic variation of Y from 0 to T which corresponds to the total realized variance on S over the period. At maturity T, the variance swap contract on S pays to the party with the long position

$$N \times \left(\frac{1}{T}[Y]_T - K_{var}\right), \quad (15)$$

where N is the notional amount of the contract and K_{var} is the variance strike of the variance swap (note that K_{var} is expressed as an annualized variance and not a total variance). $[Y]_T$ is a $\mathcal{F}_T$-measurable random variable, the value of which depends on the path of Y (or equivalently S). K_{var} is chosen at inception and is considered to be fair if the contract has a zero mark-to-market for parties entering it. For empirical applications, we choose as a convention $N = 10000$ USD or EUR depending upon the underlying asset denomination currency (hence, obtained values can be seen as expressed in *basis points*). At inception, the fair strike can be computed as

$$K_{var} = \frac{1}{T}\mathbb{E}^{\mathbb{Q}}[[Y]_T]. \quad (16)$$

For a variance swap traded at the variance strike K_{var} (not necessarily fair), the mark-to-market at inception for the party with the payer position (buyer of realized variance position) is equal to

$$VS_0 = e^{-rT}N\left(\frac{1}{T}\mathbb{E}^{\mathbb{Q}}[[Y]_T] - K_{var}\right). \quad (17)$$

This mark-to-market is zero if the variance swap is traded at its fair strike. Here, we essentially adopt the terminology and formalization of the seminal articles (Carr & Wu, 2009) and (Carr, Lee, & Wu, 2012). Another, more recent, reference dealing with variance swaps is Pun, Chung, and Wong (2015).

To make use of our approach to quantify model risk, we need to have efficient pricing methods to value variance swaps. For models in Group 1, it can be done by means of closed-form expressions or analytic results. All the needed expressions are available in the existing literature and those in closed-form are gathered in the supplementary materials appendix. For models in Group 2, it can also be done with analytic results, following the methodology introduced in Carr et al. (2012). For models built with Meixner and GH processes, the results were yet to be computed. They are provided in the supplementary materials appendix as well as the proofs leading to them.

4.2. Forward-start options

Forward-start options (hereafter FS options) are call and put options on the percentage return of the underlying asset and can be understood as options struck at a future date. Let $t = 0$ be the current time, T_0 the date at which the strike is set and T the maturity of the option. The length of time $T - T_0$ is named the tenor of the forward-start option. With $0 < T_0 < T < +\infty$, the payoff of the FS call is given by

$$CF_T(k, T_0, T) = N \times \left(\frac{S_T}{S_{T_0}} - k\right)^+, \quad (18)$$

where N is the notional amount of the option and k is a percentage strike (alternative definitions also exist for FS options, the one retained here corresponds to the contracts usually traded in the markets). For empirical applications, we also choose $N = 10000$ USD or EUR depending upon the underlying asset denomination currency. For $t \in [0, T_0[$ the FS option is a path-dependent derivative. For $t \in [T_0, T[$, S_{T_0} is known and the FS option is a vanilla option on the percentage return. The exercise decision is taken at time T. From a management standpoint, the FS option has to be handled as an exotic derivative up to T_0 and from T_0 it can be handled as a vanilla derivative and eventually merged with the vanilla options trading book.

A time 0 price for the FS call can be expressed as

$$CF_0(k, T_0, T) = Ne^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_T}{S_{T_0}} - k\right)^+\right]. \quad (19)$$

In a framework with stochastic volatility (i.e. non-stationary increments for S), the computation of the risk-neutral expectation cannot be reduced to the computation performed for a vanilla option with time to maturity $T - T_0$. One way to obtain analytical pricing results for FS options in a stochastic volatility framework is to rely on the FCE approach. This approach has been introduced for the Heston model by Hong (2004) and Kruse and Nogel (2005). Working with the FCE approach has the advantage of allowing the reliance on FFT methods for option pricing, such as the method of Carr and Madan (1999). Hence, the FCE approach is highly efficient from a numerical standpoint.

The computation of the expectation using a FFT method, requires the knowledge of the FCE. For models in our set, the expressions of the FCE are gathered in the supplementary materials appendix (Group 1) or obtained using decomposition (14) (Group 2).

5. Investigation of model risk

In this section we first detail the formal approach of model risk retained in this paper. It lies within the financial framework of Section 2 and it is based on the set of models described in Section 3. Our focus is then to explore the main aspects of model risk for variance swaps and forward-start options. For variance swaps, we study the model risk affecting a position's value. For forward-start options we study the model risk affecting a position's value and its reduction by using variance swap quotes to further constrain model calibration (Step 2 of the pricing process). Additional investigations are conducted on subgroups of the initial set of models.

5.1. The notion of model risk

An early reference dealing with model risk associated to derivative transactions is Derman (1996). Since then, numerous articles have dealt with model risk. For instance, Hull and Suo (2002) assess model risk by comparing 3 different models (stochastic volatility, Black–Scholes and local volatility). Green and Figlewski (1999) focus only on the Black–Scholes model and compare the performance of various estimators for the volatility. Kerkhof, Melenberg, and Schumacher (2010) span various risk measures and quantify market, estimation and misspecification risk on broad time series such as the S&P 500 and the U.S.dollar versus pound sterling foreign exchange rate. Gençay and Gibson (2007) survey five models and evaluate the pricing discrepancies that they imply for European options. Poulsen, Schenk-Hoppe, and Ewald (2009) study the impact of five hedging strategies on the management (P&L) of European options. Hirsa, Courtadon, and Madan (2003) and Nalholm and Poulsen (2006) provide similar insights, but for the hedging of barrier options. Detering and Packham (2016) consider potential trading losses due to model misspecifications. Barrieu and Scandolo (2015) deal with various ways to assess the model risk affecting financial risk measures.

In this paper we work under the setup developed in Cont (2006) and in which model risk is inserted in the framework of coherent risk measures of Artzner, Delbaen, Eber, and Heath (1999). Cont and Deguest (2013) work under the same setup to analyze the model uncertainty affecting multi-asset derivative prices.

Let Z be a contingent claim written on the asset S, with maturity T and payoff function z. Z can be either path-dependent or European. Let $\mathcal{M}$ be a set of models for the dynamics of S. We consider that each model in $\mathcal{M}$ is calibrated to a set of observed market quotes denoted by $\mathcal{C}$ (Step 2 of the pricing process). Basically, $\mathcal{M}$ relates to Step 1 of the pricing process and $\mathcal{C}$ to Step 2. In our study, changes in $\mathcal{C}$ are only meant to assess the robustness or sensitivity of our results to changes in Step 2.

At inception and under the choice of the model $M \in \mathcal{M}$, the market value for the party with a buyer position in Z is computed as

$$Z_0^M = e^{-rT}\mathbb{E}^{\mathbb{Q}_M}[Z_T],$$

where the superscript M means that the model M has been chosen in $\mathcal{M}$ and $\mathbb{Q}_M$ denotes the measure related to the calibrated model M (i.e. after Step 2).

As soon as there are more than one model in $\mathcal{M}$, the computation of the market value Z_0 embeds model uncertainty. This uncertainty can be understood as a risk because the wrong choice of a model can lead to a loss for the party holding a position in Z. This risk can be measured as the maximum error made by picking the wrong model out of the set $\mathcal{M}$. This definition corresponds to the approach of model risk devised in (Cont, 2006) and leads to a measure that is coherent in the sense of Artzner et al. (1999).[2] This measure of model risk is denominated in monetary units (dollars or euros) and can be written as

$$\begin{aligned}\varrho(Z_0) &= \sup_{M\in\mathcal{M}}\left\{Z_0^M\right\} - \inf_{M\in\mathcal{M}}\left\{Z_0^M\right\}\\ &= e^{-rT}\left(\sup_{M\in\mathcal{M}}\left\{\mathbb{E}^{\mathbb{Q}_M}[Z_T]\right\} - \inf_{M\in\mathcal{M}}\left\{\mathbb{E}^{\mathbb{Q}_M}[Z_T]\right\}\right).\end{aligned}$$

[2] Based on the same idea, other approaches are available for the measurement of model risk. In particular, relative model risk measures can be used. To work with these measures, one has to choose a reference model (prior model) with respect to which the measure will be defined. It appears that such risk measures are quite sensitive to the choice of the reference model. For more details on relative model risk measure, we refer to Barrieu and Scandolo (2015).

In the present paper, the model risk measure ϱ is, of course, sensitive to the choice of the set of models $\mathcal{M}$ and also to the set of calibration instruments $\mathcal{C}$. We discuss these issues below.

The requirement for $\mathcal{C}$ is that it should reflect the set of instruments for which quotes are available on the market and that are traded enough to be used as hedging instruments. This matches the practice of banks for the management of derivative books. A straightforward choice is when $\mathcal{C}$ consists of a collection of forward contracts and vanilla options with various strikes and maturities.

Most often, $\mathcal{M}$ is a finite set of models and the measure ϱ is sensitive to the choice of $\mathcal{M}$. A way to get rid of this sensitivity would be to replace the finite set $\mathcal{M}$ by the set of all models compatible with market data in order to get a model-free measure that is the largest measure. To the best of our knowledge, it is not applicable here because the structure of such set is far too general. We work instead with the set of 21 models detailed in Section 3, which are listed in Table 1. In order to immunize our conclusions from the sensitivity to $\mathcal{M}$, we have chosen a set of parametric models that can be considered large and encompassing a broad span of dynamics. Furthermore, as one of our purposes is to unveil the presence of model risk, working on a finite set is sufficient since we can only underestimate the model risk and not alter the conclusions.

We now give further details on how the model risk measure ϱ is computed for variance swaps and forward-start options. At inception, the market value of a variance swap traded at K_{var} is denoted by VS_0^M when model $M \in \mathcal{M}$ is chosen. The model risk measure with respect to the value of the variance swap can be written

$$\begin{aligned}\varrho(VS_0) &= \sup_{M\in\mathcal{M}}\left\{VS_0^M\right\} - \inf_{M\in\mathcal{M}}\left\{VS_0^M\right\}\\ &= \frac{N}{T}\left(\sup_{M\in\mathcal{M}}\left\{\mathbb{E}^{\mathbb{Q}_M}[[Y]_T]\right\} - \inf_{M\in\mathcal{M}}\left\{\mathbb{E}^{\mathbb{Q}_M}[[Y]_T]\right\}\right).\end{aligned} \quad (20)$$

This expression corresponds to a simple position in the variance swap contract. Another case of interest is when the variance swap position has been hedged using a collection of out-of-the-money (OTM) options according to the static replication of the log-contract (see the supplementary materials appendix). We name HLC_0 the value (or composition), at inception, of this hedging strip of options. Its expression is written as

$$HLC_0 = N\left(-2(r-d) + \frac{e^{-rT}}{T}\left[\int_0^{F_0}\frac{P_0(K,T)}{K^2}dK + \int_{F_0}^{\infty}\frac{C_0(K,T)}{K^2}dK\right]\right).$$

A position in the variance swap combined to its hedge in options is named HVS and its value is computed as $HVS_0 = VS_0 - HLC_0$. The model risk associated to this hedged position is equal to

$$\begin{aligned}\varrho(HVS_0) = \frac{N}{T}\Bigg(&\sup_{M\in\mathcal{M}}\Bigg\{\mathbb{E}^{\mathbb{Q}_M}[[Y]_T] - e^{-rT}\\ &\times\left[\int_0^{F_0}\frac{P_0^M(K,T)}{K^2}dK + \int_{F_0}^{\infty}\frac{C_0^M(K,T)}{K^2}dK\right]\Bigg\}\\ &-\inf_{M\in\mathcal{M}}\Bigg\{\mathbb{E}^{\mathbb{Q}_M}[[Y]_T] - e^{-rT}\\ &\times\left[\int_0^{F_0}\frac{P_0^M(K,T)}{K^2}dK + \int_{F_0}^{\infty}\frac{C_0^M(K,T)}{K^2}dK\right]\Bigg\}\Bigg),\end{aligned} \quad (21)$$

where $C_0^M(K,T)$ and $P_0^M(K,T)$ denote the prices of OTM vanilla options obtained when using model M. When the trajectories of the underlying asset price are continuous, this measure of model risk is equal to zero.

For a forward-start call option with maturity T, tenor $T - T_0$ and relative strike k, the model risk is equal to

$$\varrho(FS_0) = e^{-rT} N\left(\sup_{M\in\mathcal{M}} \left\{ \mathbb{E}^{\mathbb{Q}_M}\left[\left(\frac{S_T}{S_{T_0}} - k \right)^+ \right] \right\} - \inf_{M\in\mathcal{M}} \left\{ \mathbb{E}^{\mathbb{Q}_M}\left[\left(\frac{S_T}{S_{T_0}} - k \right)^+ \right] \right\} \right). \quad (22)$$

5.2. Data and model calibration

In the empirical part of the paper, we consider two major equity indices: the Eurostoxx 50 index (SX5E), denominated in EUR, that comprises 50 blue-chip stocks of the Eurozone and Standard and Poor's 500 index (SPX), denominated in USD, that comprises 500 stocks corresponding to the largest US firms in terms of market capitalization. These indices have been chosen because they are among the most popular underlying assets for derivative transactions traded by banks. For each index, we use two sets of market data. As is commonly accepted and widely implemented on trading desks in financial institutions across the world, we calibrate our models on option prices (see also Chapter 13 in (Cont & Tankov, 2004)). This corresponds to Step 2 of the pricing process and we detail it below.

The estimation of the models requires market data. Each dataset corresponds to a cross-section of spot, forward contracts and options closing prices taken on a given date. We have chosen two dates as representatives of different market situations. The first date is February 20, 2008 and can be considered as taken during the financial crisis period. The second date is December 19, 2012 and can be seen as a rather "*back to normal*" market situation, at least from the standpoint of market prices. Options on SPX and SX5E indices are traded and quoted on the CBOE (Chicago Board of Options Exchange) and Eurex markets, respectively. Spot, forward contracts and options closing prices as well as dividends and interest rates data were obtained from Bloomberg and Datastream. For these datasets, we also use market quotes of variance swaps on the two considered indices and for various maturities. These quotes were obtained from a major market maker operating in the interbank market.

In order to measure model risk using the approach devised in Section 5.1, we have to fit models in our set $\mathcal{M}$ to these market data (Step 2 of the pricing process). For each dataset, we calibrate these models by minimizing the sum of squared errors between model and observed prices of vanilla options. The algorithm used is a global optimization algorithm. For a given dataset and model $M \in \mathcal{M}$, the calibrated model parameter $\theta_M^{*,1}$ is obtained as

$$\theta_M^{*,1} = \arg\min_{\theta_M\in\Theta_M} \sum_{i=1}^{N_T}\sum_{j=1}^{N_K} \left(O(K_j, T_i; \theta_M) - O^{Obs}(K_j, T_i) \right)^2, \quad (23)$$

where Θ_M is the set of feasible parameters for model M, N_T the number of maturities in the options set, N_K the number of strikes for each maturity (without loss of generality we consider the same number of strikes is available for each maturity). $O(.; \theta_M)$ denotes the option price obtained using the chosen model with parameter θ_M and $O^{Obs}(.)$ denotes the corresponding observed price. An option $O(.)$ in the set can be a call or a put. In our datasets we work with eight maturities ranging from one month to five years (hence $N_T = 8$) and nine strikes for each maturity that are specified in terms of moneyness[3]. More specifically, these strikes are 60, 80, 90, 95, 100, 105, 110, 120 and 150 percent (hence $N_K = 9$). This set of calibration instruments is denoted by $\mathcal{C}_1$ and corresponds to the usual set used to calibrate models to market data.

In order to assess the robustness of our results to the estimation step, we also consider a broader set of calibration instruments. This set, $\mathcal{C}_2$, consists of $\mathcal{C}_1$ plus a set of variance swaps with the same maturities as vanilla options (hence, eight quotes). When a model is calibrated to $\mathcal{C}_2$, we minimize, at the same time, the sum of squared errors between model and observed prices of options and variance swaps. For a given dataset and model $M \in \mathcal{M}$, the calibrated model parameter $\theta_M^{*,2}$ is obtained as

$$\theta_M^{*,2} = \arg\min_{\theta_M\in\Theta_M} \left\{ \sum_{i=1}^{N_T}\sum_{j=1}^{N_K} \left(O(K_j, T_i; \theta_M) - O^{Obs}(K_j, T_i) \right)^2 + \sum_{i=1}^{N_T} \left(VS(T_i; \theta_M) - VS^{Obs}(T_i) \right)^2 \right\}, \quad (24)$$

where $VS(.; \theta_M)$ denotes the variance swap price obtained using the chosen model with parameter θ_M and $VS^{Obs}(.)$ denotes the corresponding observed price.

Once the minimization programs have been solved for each model, it is possible to measure the quality of the calibration. For a given calibrated model, the fit quality can be measured as mean absolute error (MAE) and root mean squared error (RMSE) on prices or on the associated implied volatilities. When applied to implied volatilities, error measures are written (dropping the reference to the chosen model) as

$$MAE = \sum_{i=1}^{N_T}\sum_{j=1}^{N_K} \frac{\left| \sigma(K_j, T_i; \theta^*) - \sigma^{Obs}(K_j, T_i) \right|}{N_K N_T}, \quad (25)$$

$$RMSE = \sqrt{ \sum_{i=1}^{N_T}\sum_{j=1}^{N_K} \frac{\left(\sigma(K_j, T_i; \theta^*) - \sigma^{Obs}(K_j, T_i) \right)^2}{N_K N_T} }, \quad (26)$$

where $\sigma(K_j, T_i; \theta^*)$ and $\sigma^{Obs}(K_j, T_i)$ are the implied volatilities[4] corresponding to the option prices $O(K_j, T_i; \theta^*)$ and $O^{Obs}(K_j, T_i)$, respectively. These metrics are expressed in terms of volatility. The models in $\mathcal{M}$ provide a proper fit, for each dataset, to $\mathcal{C}_1$ and $\mathcal{C}_2$. The obtained MAE and RMSE appear to be both around three quarters of a volatility point for options (less than half a volatility point for at-the-money options), which can be regarded as a very good fit given the strike and maturity spans of the option sets. For variance swaps in $\mathcal{C}_2$ the obtained error measures appear to be between two and eight basis points. Again, this can be considered as a very good fit given the span of maturities and the fact that the models are jointly calibrated to options and variance swaps. The estimated model parameters for the two calibration sets are available upon request, as well as the details of the mentioned error measures.

5.3. Empirical results for variance swaps and forward-start options

In Fig. 2, we plot, for each dataset, the model risk measures for variance swaps of various maturities (T), both without (Eq. (20)) and with (Eq. (21)) their hedges in OTM vanilla options. It appears clearly that model risk is present in the value of variance swaps when models are calibrated to a large sample of vanilla options. In the base case where the positions are not hedged, the model risk is above 200 basis points for the smallest maturity in the four

[3] The moneyness is equal to the strike divided by the corresponding spot price of the index according to this options market convention. For instance, a moneyness of 80 percent means that the strike is equal to 80 percent of the current value of the index.

[4] The implied volatility of a given option price is the volatility that needs to be plugged in the Black–Scholes formula to obtain this price. It can be seen as a measure of the future volatility of the underlying assessed by the agents on the market.

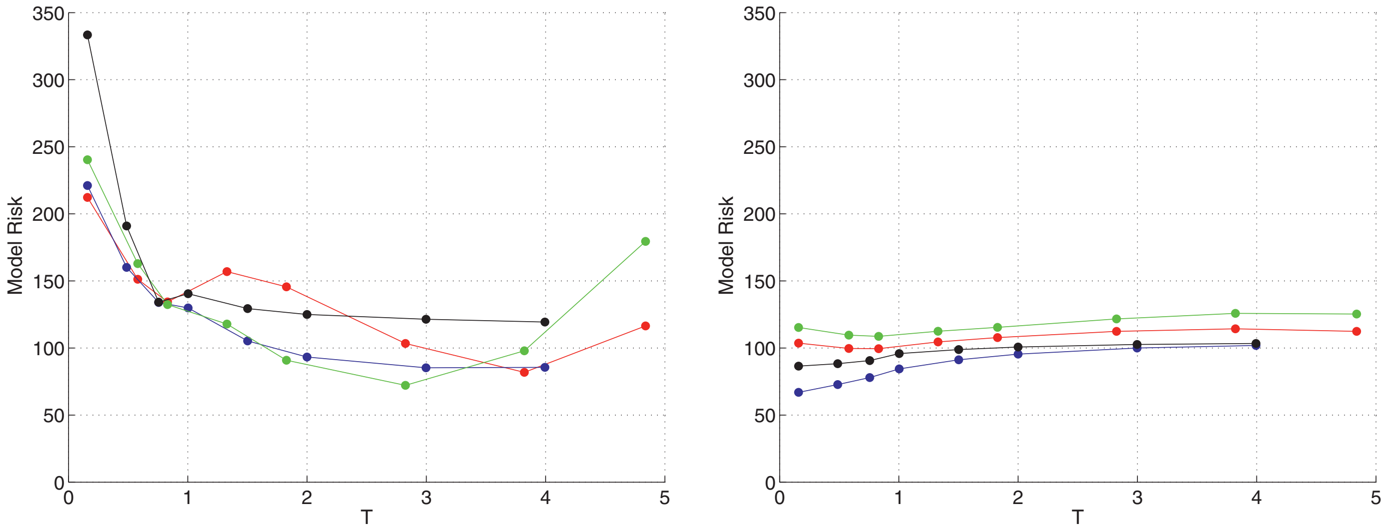

Fig. 2. Measures of model risk for variance swaps on SPX and SX5E, with (Eq. (20)) and without (Eq. (21)) their hedge in OTM options (right and left panels, respectively). The variance notionals are $N = 10000$ USD or EUR. *Red, blue, green* and *black* lines are respectively for SPX-2008, SPX-2012, SX5E-2008 and SX5E-2012. T is the maturity of the variance swaps. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

configurations we have tested. For maturities beyond 1 year, the model risk fluctuates around 100 basis points. We further observe that model risk is more stable across maturities when the VS are hedged. In this case, the dispersion of model risk is reduced, but remains centered around 100 basis points. This is surprising because we would have expected the hedge to strongly mitigate the model risk. Moreover, the computation of model risk for variance swaps combined to their traditional hedging portfolio allows us to disentangle the measure from potential differences in the value of the hedging portfolio. It allows us to focus on the model risk stemming from variance swaps themselves.

Accordingly, even after accounting for potential discrepancies in the value of their hedging portfolio, model risk remains pervasive in variance swap prices. Hence, we can say that variance swaps are genuine exotic instruments in the sense that model risk exists even when models are calibrated to the available data for vanilla options. This is further confirmed by the fact that model risk remains even when the popular hedge in OTM options is implemented. This evidence of the exotic feature of variance swap prices should have implications for their management by financial institutions.

This evidence of model risk can also be understood as information about the risk-neutral dynamics of the underlying asset price carried by variance swap quotes when available. Adding variance swaps to the set of calibration instruments containing only options appears a good idea as these instruments carry a sizeable amount of additional information. With this new calibration set, denoted by $\mathcal{C}_2$, the model risk of the price of another exotic derivative should be smaller compared to the model risk when models are calibrated to $\mathcal{C}_1$. To investigate this intuition, we have chosen to work with forward-start options because the price of these products is highly sensitive to the underlying dynamics of the chosen model and, as such, have the reputation to be difficult to handle among practitioners. Hence reducing the model risk stemming from these products is of interest for financial institutions using them.

Fig. 3 plots, for each dataset and for several maturities (T) between 1 and 3 years, the model risk measures of 1-year tenor forward-start call options struck in, at and out of the money ($k = 0.75, 1.00$ and 1.25, respectively). These results are obtained when models are calibrated on instruments in $\mathcal{C}_1$ (left column) and in $\mathcal{C}_2$ (right column).

Similar to the case of variance swaps, we acknowledge non-negligible model risk for forward-start options. A new feature is that model risk seems to be increasing in maturity for forward-start options with the same tenor. It means that the model risk associated to a forward-start option with a longer maturity is greater than model risk for a forward-start option with shorter maturity. Moreover, we note that calibration to instruments in $\mathcal{C}_2$ leads to an overall lower model risk than the calibration to $\mathcal{C}_1$ (this is always verified when model risk is above 100 basis points). The addition of variance swaps to the set of calibration instruments is therefore fruitful. Furthermore, we note that there is no particular ordering of model risk according to moneyness: depending on the underlying and market conditions, in-the-money (ITM, $k = 0.75$) options may be more or less subject to model risk, compared to their OTM counterparts. For instance, we see that model risk was higher for OTM options in 2008 compared to 2012 (red and green curves above the blue and black ones), but this statement does not hold for ITM options. We believe that these final comments can help practitioners value and handle their forward-start option portfolios.

5.4. Model risk of subgroups

With 21 models at our disposal, one may wonder which natural subgroups of this set are likely to explain the largest proportion of model risk. In order to answer this question, we have computed the same measure of model risk over 11 subgroups of our initial set. We define the Average Proportion of Model Risk (*APMR*) captured by these subgroups as

$$APMR^j = \frac{1}{N}\sum_{i=1}^{N} \frac{\varrho_i^j}{\varrho_i^{total}}, \tag{27}$$

where j is the index of the chosen subgroup and i is the index of the derivative transaction that is priced on a given date and market. N denotes the total number of such transactions. ϱ_i^j is the model risk of derivative i with respect to subgroup j, while ϱ_i^{total} is the corresponding total model risk computed over the 21 models. We repeat this exercise for simple variance swap transactions (VS), for hedged variance swaps (VSH), for forward-start options when models are calibrated to the set $\mathcal{C}_1$ (FSO1) and to the set $\mathcal{C}_2$ (FSO2).

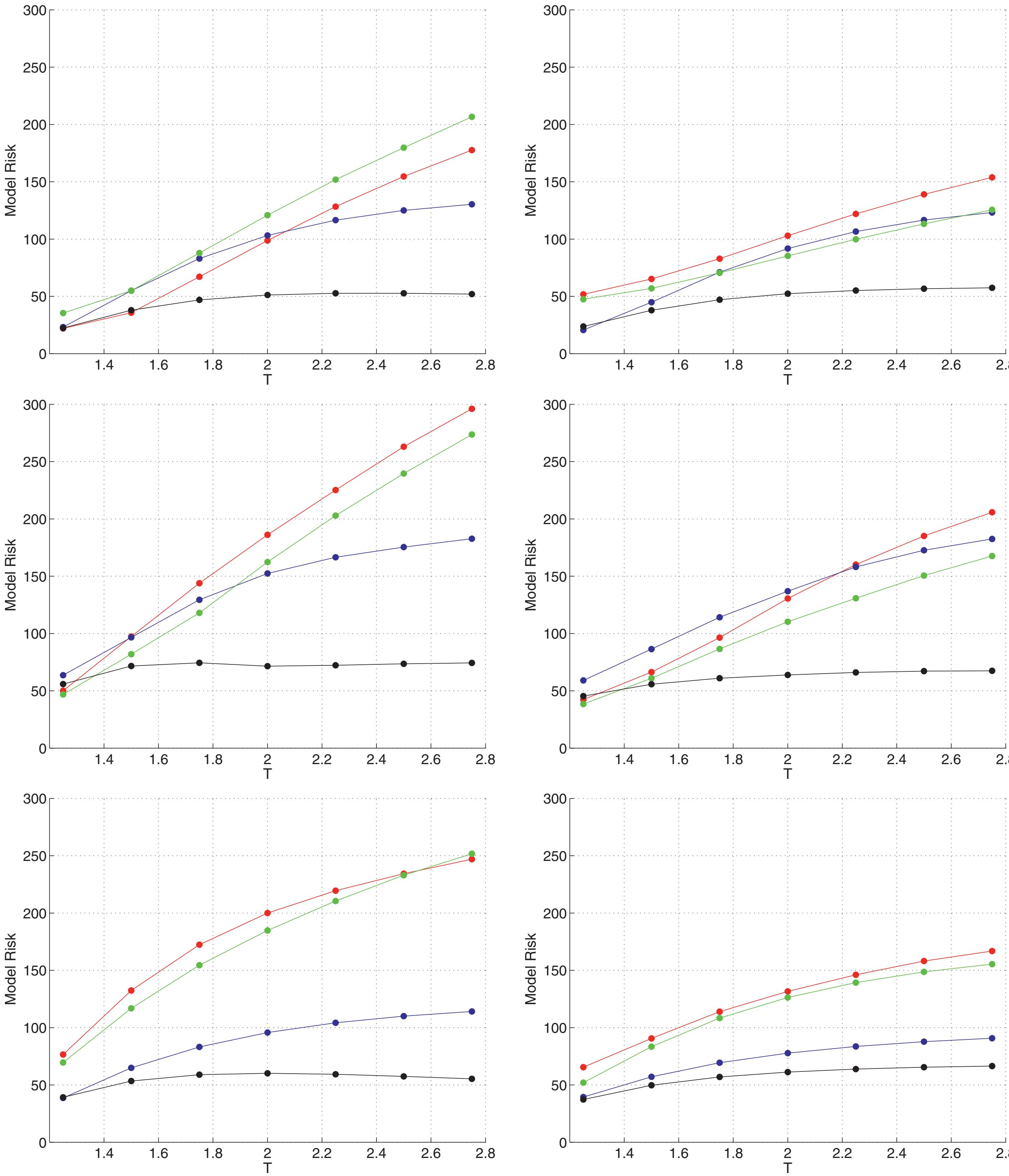

Fig. 3. Measures of model risk for forward-start call options (Eq. (22)) on SPX and SX5E, with $N = 10000$ USD or EUR, tenor is 1 year and $k = 0.75$ (top), $k = 1.00$ (center) and $k = 1.25$ (bottom). Models are calibrated to vanilla options only (*Left panel*) or jointly calibrated to vanilla options and variance swap quotes (*Right panel*). *Red, blue, green* and *black* lines are respectively for SPX-2008, SPX-2012, SX5E-2008 and SX5E-2012. T is the maturity of the forward-start option. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 4
Average Proportion of Model Risk (APMR) captured by subgroups. It is computed using formula (27) over N derivatives (second line of the table). The number of models within each subgroup is indicated in the second column.

	Nb models	VS	VSH	FSO1	FSO2	All
Nb derivatives (N)		32	32	84	84	232
Group 1	3	0.267	0.055	0.199	0.231	0.200
Group 2	18	0.871	0.901	0.453	0.479	0.582
ICIR	6	0.301	0.370	0.233	0.320	0.293
OUG	6	0.659	0.601	0.263	0.322	0.385
OUIG	6	0.285	0.269	0.184	0.302	0.252
VG	3	0.300	0.231	0.246	0.214	0.240
CGMY	3	0.615	0.677	0.270	0.237	0.362
KOU	3	0.242	0.185	0.224	0.196	0.211
MEIX	3	0.284	0.287	0.255	0.196	0.242
NIG	3	0.398	0.454	0.285	0.207	0.296
GH	3	0.452	0.472	0.226	0.183	0.275

We make three distinctions. The first is between the two groups (1 and 2) of models: models belonging to the Heston family *versus* models based on Lévy processes. The second distinction is among the models of Group 2: we set the stochastic clocks (ICIR, OUG and OUIG) and look at model risk across the driving Lévy processes. Lastly, the last distinction is the opposite: we set the driving Lévy process and compute the model risk across the time changes. The values of APMR defined in (27) are gathered in Table 4.

A simple first conclusion is that the APMR is overall larger for variance swaps than for forward-start options. This is true for all subgroups, but especially for Group 2, OUG and CGMY, even though the reasons behind these larger discrepancies remain unclear.

Quite logically, the highest APMR are obtained for Group 2, which is by far the largest subset. However, we observe that over all priced derivatives, the models based on Lévy processes only explain 58.2 percent of model risk on average. This is a lot more than the 20 percent explained by the models belonging to the Heston family, but it is clear that the sum of both model risk measures is, on average lower than the total model risk. As such, the presence of several types of dynamics within the set of models appears to be crucial.

A second order conclusion is that the stochastic clock and the driving Lévy process seem to have a comparable importance in the determination of model risk. This finding means that a similar care should be devoted to the choice of each component. Two subgroups in each category lead to much higher average model risk (OUG and CGMY). We leave it for future research to eventually unveil the theoretical grounding of these findings.

6. Conclusion

This paper considers the problem of model risk for variance swaps and forward-start options that arises in a realistic market setup where the underlying asset price process exhibits stochastic volatility and jumps. Our findings evidence the exotic feature of variance swaps. The model risk affecting their value is estimated around *100 basis points* irrespectively of their maturity. The presence of model risk is not affected by the application of the popular hedge with a strip of OTM options. Forward-start options are found to embed model risk as well. This model risk is found to increase with the maturity. We find that adding variance swaps in the set of calibration instruments generates a reduction of model risk. En route, we provide some novel closed-form pricing formulae for variance swaps and forward-start options. Lastly, we show that the set of models with which model risk is computed must include several dynamics based on continuous and jump processes. We also find that, within the group of Lévy based models, the choice of the driving Lévy process embeds approximately as much model risk as the choice of the stochastic clock. The conclusions of our investigations as well as the developed framework can favor the understanding and measurement of risks taken by financial institutions such as banks. Hence, our paper contributes to a better risk management by banks. They can, in turn, make better decisions concerning the allocation of their available capital. Our contributions also permit a better confidence in risk statements issued by banks and eventually allow their control.

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Supplementary material

Supplementary material associated with this article can be found, in the online version, at 10.1016/j.ejor.2016.03.018.

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